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Fractional Indexes Impulse Responses Approximation for Discrete-Time Weyl Symbol Computation

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Introduction

Linear systems are of fundamental importance in many engineering applications. Linear time-varying systems in comparison to linear time-invariant ones have important advantages when the signals are modelled or processed nonstationary [1]. Moreover linear time-varying systems are powerful tools for processing nonstationary random processes. On the other hand quantum physics and time-frequency signal theory are strongly interrelated fields. The phase space of physics (momentum and position) plays similar role in quantum mechanics as the time-frequency plane in signal theory (time and frequency) [2].

In order to describe dynamics of time-varying discrete-time systems one can use difference equations with time-dependent coefficients or a generalized description employing state equations with time-dependent matrices

$$\mathbf{x}(k+1)=\mathbf{A}(k)\mathbf{x}(k)+\mathbf{B}(k)\mathbf{v}(k), \quad (1)$$

$$\mathbf{y}(k)=\mathbf{C}(k)\mathbf{x}(k)+\mathbf{D}(k)\mathbf{v}(k), \quad (2)$$

where $\mathbf{A}(k)\in {}^{n_k\times n_k}$, $\mathbf{B}(k)\in {}^{n_k\times m_k}$, $\mathbf{C}(k)\in {}^{p_k\times n_k}$, $\mathbf{D}(k)\in {}^{p_k\times m_k}$, $\mathbf{x}(k)\in \mathsf{R}^{n_k}$, $\mathbf{v}(k)\in \mathsf{R}^{m_k}$, $\mathbf{y}(k)\in \mathsf{R}^{p_k}$, $k\in \mathsf{Z}$.

Definition. State dimension sequence is defined for each time sample and can be given in following way

$$\mathbf{n}=\left\{n_k : k\in \;, n_k\in \;, \mathbf{x}(k)\in {}^{n_k}\right\}= \\ =\left[\ldots, n(k-1), n(k), n(k+1), n(k+2), \ldots\right] \quad (3)$$

Input and output dimension sequences are defined by analogy.

Alternatively the description can be converted thanks to general linear operators theory [3–6] and their applications for discrete-time, time-varying systems [7, 8] into a more general equation operator description.

Time-Frequency Transform

Frequency methods are well known tools applicable not only for linear time-invariant systems, but also for linear time-varying systems with [9–12].

Time-frequency methods are known for more than 60 years [13]. Many investigations has been made until now e.g. [1, 2, 14–18]. The time-frequency transform is formulated as parameterized extension of Laplace transform and it is defined for continuous time systems

$$Z_{\hat{\mathbf{K}}}(t,f)=\left.\frac{(\hat{\mathbf{K}}v)(t)}{v(t)}\right|_{v=e^{j2\pi ft}}. \quad (4)$$

General form of the transform for continuous time systems can be defined by following Generalised Weyl Symbol [15]

$$L^{(\alpha)}(t,f)=\int_{\tau} h\left(t+\left(\tfrac{1}{2}-\alpha\right)\tau, t-\left(\tfrac{1}{2}+\alpha\right)\tau\right)e^{-j2\pi f\tau}d\tau. \quad (5)$$

where $\alpha\in \mathsf{R}$ is arbitrary real number, usually bounded such that $|\alpha|\le 0.5$ and $h(t_1,t_0)$ is system response at time t_1 for shifted by time t_0 Dirac impulse $\delta(t-t_0)$.

Depending on the parameter α we can distinguish 3 following forms of the time-frequency transformation:

- Time-varying Zadeh transfer function [13] for α=0.5,
- Frequency dependent modulation function [14] or Kohn-Nirenberg symbol [19] for α=-0.5,
- Weyl symbol, taken from quantum mechanics [20] for α=0.

Time-Frequency Transform in Discrete-Time

Equation (5) cannot be employed directly for discrete-time. It must be converted e.g. using the Discrete Fourier

Transform (DFT)

$$L_{\hat{\mathbf{T}}}^{(\alpha)}\left(t_k, f_l\right)=\sum_{n=1}^{N} h\left(k+\left(\tfrac{1}{2}-\alpha\right)n, k-\left(\tfrac{1}{2}+\alpha\right)n\right)e^{-j2\pi ln/N} . \quad (6)$$

where $t_k = kT_p, f_l = \dfrac{l}{NT_p},\ l = 0,1,\ldots,{}^{N}\!/_{2}-1,\ k = 0,1,\ldots,N.$

For linear time-invariant systems on infinite time horizon the time-frequency transform holds following property

$$\underset{k,m}{\forall} L_{\hat{\mathbf{T}}}^{(\alpha)}\left(t_k, f_l\right)=L_{\hat{\mathbf{T}}}^{(\alpha)}\left(t_m, f_l\right) . \quad (7)$$

It mean that the Generalised Weyl Symbol for LTI systems is time invariant.

Discrete-time Kohn-Nirenberg symbol (for α=-0.5) can be written in following form

$$B_{\hat{\mathbf{T}}}\left(t_k, f_l\right)=L_{\hat{\mathbf{T}}}^{(-0.5)}\left(t_k, f_l\right)=\sum_{n=1}^{N} h\left(k+n, k\right)e^{-j2\pi ln/N} .(8)$$

For arbitrary given k the Kohn-Nirenberg symbol is computed from present and future system responses. Discrete-time, time-varying Zadeh transfer function (for α=0.5) can be written in following form

$$Z_{\hat{\mathbf{T}}}\left(t_k, f_l\right)=L_{\hat{\mathbf{T}}}^{(0.5)}\left(t_k, f_l\right)=\sum_{n=1}^{N} h\left(k, k-n\right)e^{-j2\pi ln/N} .(9)$$

For arbitrary given k the time-varying Zadeh transfer function is computed from present and past system input – requires knowledge about the past from current k. Discrete-time analogy to the Weyl symbol (for α=0) can be written either in the form known in the literature [21]

$$L_{\hat{\mathbf{T}}\infty}^{(0)}\left(t_k, f_l\right)=2\sum_{n=-\infty}^{\infty} h\left(k+n, k-n\right)e^{-j4\pi ln/N} . \quad (10)$$

For arbitrary given k Weyl symbol is computed from present, future and past system inputs and outputs – requires knowledge about the past and future from current k.

Above definition needs two times larger time horizon in order to ensure the same accuracy as Kohn-Nirenberg symbol and time-varying Zadeh transfer function. The main reason is the multiplier 2 in (10). It mean that each value of impulse response is taken 2 times.

Let us consider SISO system defined on finite time horizon of length N and k equal to 0. The Kohn-Nirenberg symbol is calculated on the basis of N element vector $\left[h(0,0), h(1,0), \ldots, h(N-1,0)\right]$ which corresponds to resolution in: discrete frequency of 1/N.

Discrete-time, time-varying Zadeh transfer function calculated for the same conditions has also N element vector basis $\left[h(0,0), h(0,-1), \ldots, h(0,1-N)\right]$. The resolution in frequency domain is identical as for Kohn-Nirenberg symbol.

Discrete-time Weyl symbol from eq. (10) calculated also for k=0 and time horizon of length N has two times shorter basis (N/2+1) for odd N $\left[h(0,0), h(1,-1), h(2,-2), \ldots, h\left(\tfrac{N-1}{2}, -\tfrac{N-1}{2}\right)\right]$. It follows to deterioration of the resolution in frequency domain (two times larger).

The main reason of the deterioration is the fact that using definition (10) large amount of information given in the impulse response $h(k_1, k_2)$ is neglected. Especially the formula (10) does not take into computations impulse responses with odd differences $k_1 - k_2$. See for example two bases for k=0 and k=1 and unused elements of impulse response

$$\begin{array}{l} k=0 \\ \text{unused} \\ k=1 \end{array} \begin{bmatrix} h(0,0) & & h(1,-1) & \cdots & h\left(\tfrac{N-1}{2}, \tfrac{-N+1}{2}\right) \\ & \cancel{h(1,0)} & & \cdots & \\ h(1,1) & & h(2,0) & \cdots & h\left(\tfrac{N+1}{2}, \tfrac{-N+3}{2}\right) \end{bmatrix} .$$

Novel computation algorithm for Discrete-Time analogy to Weyl symbol

Discrete-time computational algorithm of the Weyl symbol given in [21] can be improved by taking into consideration full information included in the system impulse response $h(k_1, k_2)$.

In practise discrete time moments with fractional indexes are directly infeasible, but they can be interpolated from neighbour discrete time responses, see for example

$$\begin{bmatrix} h(1,0) & & h(2,0) \\ & h(1.5, 0.5) & \\ h(1,1) & & h(2,1) \end{bmatrix} . \quad (11)$$

In that way responses with both even and odd differences $k_1 - k_2$ are taken account in computation.

In order to take account impulse responses with odd differences $k_1 - k_2$ fractional indexes elements will be defined first.

Definition. Fractional index discrete time response value of one variable $h(k+0.5),\ k \in \mathsf{Z}$ is defined as linear interpolation of h taken in following way

$$h(k+0.5)=0.5\left(h(k)+h(k+1)\right) . \quad (12)$$

Definition. Fractional index discrete time response value of two variables $h(k+0.5, l+0.5),\ k,l \in \mathsf{Z}$ is defined as 2-D linear interpolation of h taken in following way

$$\begin{aligned} & h(k+0.5, l+0.5)= \\ & =0.25\left(h(k,l)+h(k+1,l)+h(k,l+1)+h(k+1,l+1)\right) . \quad (13) \end{aligned}$$

Taking account eq. (6) following substitution holds

$$\begin{cases} a=k+\left(\tfrac{1}{2}-\alpha\right)n,\ \underline{a}=\operatorname{floor}(a),\ \overline{a}=\underline{a}+1, \\ b=k+\left(\tfrac{1}{2}+\alpha\right)n,\ \underline{b}=\operatorname{floor}(b),\ \overline{b}=\underline{b}+1, \end{cases} \quad (14)$$

where under-bar (floor) denotes round toward minus infinity.

Main difference can be seen in fig. 1 on example with two impulse responses generated for the same system. The system is low-pass FIR Butterworth 4th order filter with cut-off frequency $\Omega_c = 0.2$. Integer indexes response mean the response $h(k+n, k-n)$, whereas fractional indexes response mean response $h(k+0.5n, k-0.5n)$. In both cases k is constant and $n = 0,1,2,...,N-1$.

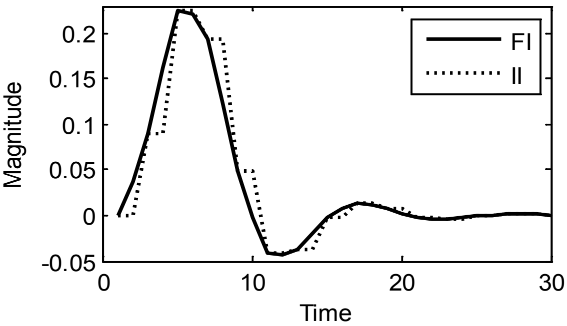

Fig. 1. Selected impulse responses for fractional indexes (FI – solid line) and integer indexes (II – dotted line) approximation

Having defined impulse responses with fractional indexes the discrete-time Weyl symbol can be adopted directly from (6)

$$L_{\hat{\mathbf{T}}}(t_k, f_l) = L_{\hat{\mathbf{T}}}^{(0)}(t_k, f_l) = \sum_{n=1}^{N} h(k+0.5n, k-0.5n) e^{-j2\pi ln/N}. \quad (15)$$

Discrete-time Weyl symbol from eq. (15) calculated for k=0 and time horizon of length N has following basis $\left[h(0,0), h(0.5,-0.5), h(1,-1), ..., h\left(\frac{N-1}{2}, -\frac{N-1}{2}\right)\right]$ with length N (for odd N).

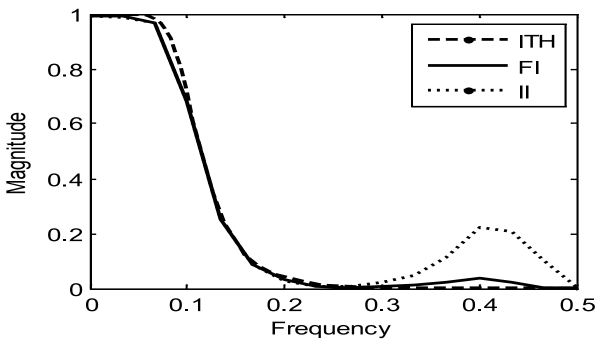

Fig. 2. Magnitude-frequency diagrams for fractional indexes (FI – solid line), integer indexes (II – dotted line) approximation and infinite horizon response (ITH – dashed line) for the same system

Taking account interpolation defined by eq. (13), the second, fractional index term in the basis is equal to: $h(0.5,-0.5) = 0.25\left(h(0,-1) + h(1,-1) + h(0,0) + h(1,0)\right)$. In this case the resolution in frequency domain is $1/N$.

Differences in frequency domain can be seen in fig. 2. The figure shows 3 different frequency diagrams calculated for fixed k. Dashed line (ITH) shows classical magnitude-frequncy spectra of the low-pass filter defined on infinite time horizon. Dotted and solid lines show diagrams calculated for finite time horizon of N=30 steps. Dotted line draw frequency response for integer indexes impulse response (10) whereas solid line draw frequency response for fractional indexes response (15). Taking the dashed line as the reference, significant improvement can be seen between dotted line – far from the dahed one and the solid one – close to the reference even for short time horizon (30 steps).

Conclusions

A new concept of fractional indexes impulse response approximation for discrete-time, time-varying systems is introduced. The concept can be successfully applied for e.g. computation of Weyl Symbol as well as Generalised Weyl Symbol. Using fractional indexes approximation allow to use all elements of impulse response both with even and odd differences between first and second variable index. Resulting average accuracy is similar like integer indexes methods Kohn-Nirenberg symbol with $\alpha = -0.5$ that focuses on the beginning of the time horizon and Zadeh time-frequency transfer function with $\alpha = 0.5$ which focuses of the end of the time horizon. In contradistinction to two mentioned methods Weyl symbol $\alpha = 0$ allows to focus the highest accuracy for the middle part of the time window. Generalised Weyl symbol allow to choose other value $\alpha \in \langle -0.5, 0.5 \rangle$, that additionally allows to focus the highest accuracy on particular part of the time window.

Fractional indexes approximation allow to compute Generalised Weyl Symbol for arbitrary real α with accuracy close to methods with $\alpha = \pm 0.5$ with the same length of the time horizon.

Acknowledgements

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Time-frequency transformations including definitions of the Weyl symbol and generalised Weyl symbol are well known tools for linear time-varying systems analysis. Unfortunately known discrete-time analogy to Weyl symbol uses only half of the information contained in the discrete-time system transfer operator (system impulse responses with even indexes differences are only taken account for the computation). Moreover the method cannot be used for computation of the discrete-time analogy to generalized Weyl symbol. In the paper a novel computation approach is proposed. First of all the concept of fractional indexes for discrete-time systems is introduced. Secondly the algorithm for approximation impulse responses of the discrete-time system with fractional impulse responses is proposed. Our method can be used for computation of the generalised Weyl symbol as well as for computing Weyl symbol based on full information available in the discrete-time system transfer operator. Differences between the existing and the proposed computation method for Weyl symbol are illustrated on the numerical example in the time domain. Ill. 2, bibl. 21 (in English; abstracts in English, Russian and Lithuanian).

П. Орловски. Приближенные вычисления дискретного символа Вейля на основе дробных индексов импульсов ответа // Электроника и электротехника. – Каунас: Технология, 2010. – № 8(104). – С. 9–12.

В анализе линейных нестационарных системах широко известны определенные и обобщенные символы Вейла. Однако в таком анализе используется только половина информации и не может быть использован для вычисления дискретного времени с обобщенным символом Вейла. Предлагается оригинальный метод вычисления аппроксимации импульсных характеристик дискетных систем. Разработанный метод позволяет определить и точно найти обобщенные символы Вейла. Во временной области на численном примере иллюстрируется существующие и вновь предлагаемые методы вычисления символов Вейла. Ил. 2, библ. 21 (на английском языке; рефераты на английском, русском и литовском яз.).

P. Orlowski. Apytikris Veilo simbolių skaičiavimas naudojant atsakų trupmeninius impulsų indeksus // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2010. – Nr. 8(104). – P. 9–12.

Tiesinių stacionarių sistemų analizei plačiai naudojami apibendrinti Veilo simboliai. Šiai analizei panaudojama tik pusė gautos informacijos. Be to, jos negalima taikyti skaitmeniniams parametrams nustatyti. Siūlomas originalus analizės būdas, kai naudojami trupmeniniai impulsų indeksai. Šiam skaičiavimui sukurtas algoritmas, įvertinantis skaitmeninių sistemų impulsines charakteristikas. Pateiktas senu ir siūlomu būdu apskaičiuotų Veilo simbolių palyginimas. Il. 2, bibl. 21 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).