http://dx.doi.org/10.5755/j01.eee.20.5.7098 ELEKTRONIKA IR ELEKTROTECHNIKA, ISSN 1392–1215, VOL. 20, NO. 5, 2014

Application of Stochastic Differential-Algebraic Equations in Hybrid MTL Systems Analysis

L. Brancik[1], E. Kolarova[2]
[1]*Department of Radio Electronic, Brno University of Technology, Technicka 12, 616 00 Brno, Czech Republic*
[2]*Department of Mathematics, Brno University of Technology, Technicka 8, 616 00 Brno, Czech Republic*
brancik@feec.vutbr.cz

[1]***Abstract*—The paper deals with the application of stochastic differential-algebraic equations (SDAE) in the field of the time-domain simulation of hybrid (lumped/distributed) systems with randomly varying parameters. A core of the method lies on the theory of stochastic differential equations (SDE) considering the system responses as stochastic processes. However, due to a hybrid nature of the system, namely its lumped parameter part, non-differential (algebraic) parts arise generally in the solution. Herein, multiconductor transmission lines (MTL) play a role of the distributed-parameter parts of the hybrid system. The MTL model is designed as a cascade connection of generalized RLCG T-networks, while the state-variable method is applied for its description. The MTL boundary conditions are incorporated through a modified nodal analysis (MNA) to cover arbitrarily complex circuits. System responses are formed by the sets of stochastic trajectories completed by corresponding sample means and respective confidence intervals. To get the results a weak stochastic backward Euler scheme is used, consistent with the Itô stochastic calculus. All the computer simulations have been performed in the Matlab language environment.**

***Index Terms*—Computer simulation, stochastic processes, time-domain analysis, transmission lines.**

I. INTRODUCTION

Stochastic differential equations (SDE) approach finds its place in many fields of the engineering when stochastic changes in physical systems should be taken into account [1], [2]. In the electrical engineering it can cover a number of random processes arising in electrical circuits [3]–[5]. Interesting possibility is to apply this approach for systems with distributed parameters, namely single or multiconductor transmission lines (MTL), including those implemented on semiconductor substrates, which are exploited in high-speed mixed (analog/digital) circuits to transmit data [6]-[8]. In the paper, an attention is paid to hybrid (lumped/distributed) systems with multiconductor transmission lines (MTL) as their distributed-parameter parts, whereas MTLs' primary parameters can vary randomly. Due to the lumped-parameter parts of the system, however, a non-differential (algebraic) part in its description is generally present. That is why stochastic differential-algebraic equations (SDAE) have to be utilized in this case. Some attention has been paid to basic RLCG networks, [9], [10], which can be used as building blocks for more complex physical models, as transmission lines (TL) [6], [7]. An RLCG model of a simple TL is shown in [11], [12], where stochastic responses caused by random excitation and the model's parameters have been analysed. The multiconductor transmission lines (MTL) represented by generalized RLCG-based models were simulated in [13], with boundary conditions folded via generalized Thévenin equivalents of terminal circuits, and in [14], where modified nodal analysis (MNA) was used in this respect, enabling to consider an arbitrarily complex lumped-parameter part. A hybrid system with several MTLs in Fig. 1 will be used for further considerations, while the MTLs' models in Fig. 2 are employed. Such a system has also been simulated in [15], however, a numerical method applied in this paper differs, namely it is of a weak convergence utilizing two-point distributed random variables as an initial stochastic process, which is more convenient if sample moments are needed in the resultant stochastic process [2].

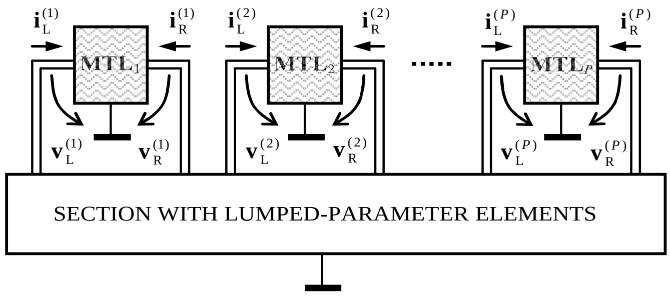


Fig. 1. Hybrid system with several MTLs.

In the paper, some MTLs per-unit-length (p.u.l.) parameters are regarded as randomly varying, which is marked by * in Fig. 2.

II. HYBRID SYSTEM BLOCK DIAGRAM

We will consider a hybrid system in Fig. 1 composed of a lumped parameter section and a number of MTLs, namely the MTL_r, $r = 1,2,...,P$.

Based on the concept in [12] the lumped-parameter section is described by the MNA method

Manuscript received November 27, 2014; accepted February 16, 2014.
This work was performed in laboratories supported by the SIX project CZ.1.05/2.1.00/03.0072 the operational program Research and Development for Innovation, and supported by the project no. FEKT-S-14-2281.

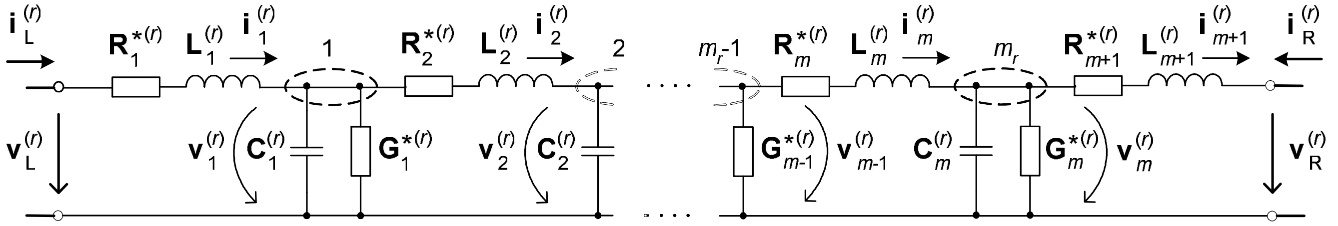


Fig. 2. Generalized T-networks model of the r-th MTL$_r$.

$$\mathbf{C}_\mathrm{N}\frac{d\mathbf{v}_\mathrm{N}(t)}{dt}+\mathbf{G}_\mathrm{N}\mathbf{v}_\mathrm{N}(t)+\sum_{r=1}^{P}\left(\mathbf{S}_\mathrm{L}^{(r)}\mathbf{i}_\mathrm{L}^{(r)}(t)+\mathbf{S}_\mathrm{R}^{(r)}\mathbf{i}_\mathrm{R}^{(r)}(t)\right)= \\ =\mathbf{i}_\mathrm{N}(t), \tag{1}$$

where $\mathbf{C}_\mathrm{N}$ and $\mathbf{G}_\mathrm{N}$ are $N \times N$ matrices with entries defined by lumped reactive and resistive components, respectively, $\mathbf{v}_\mathrm{N}(t)$ is an $N \times 1$ vector of node voltages appended by currents of independent voltage sources and inductors, $\mathbf{i}_\mathrm{N}(t)$ is an $N \times 1$ vector of the exciting source waveforms, $\mathbf{i}_\mathrm{L}^{(r)}(t)$ and $\mathbf{i}_\mathrm{R}^{(r)}(t)$ are $n_r \times 1$ vectors of currents entering the r-th MTL$_r$ from its left and right side, respectively, with n_r as the number of its active wires, $\mathbf{S}_\mathrm{L}^{(r)}$ and $\mathbf{S}_\mathrm{R}^{(r)}$ are $N \times n_r$ selector matrices, with entries $s_{ij} \in \{0,1\}$, mapping current vectors into the node space of the hybrid system, respectively. These currents satisfy the MTLs boundary conditions, and are bound with respective voltages through $\mathbf{v}_\mathrm{N}(t)$ vector by:

$$\mathbf{v}_\mathrm{L}^{(r)}(t)=\mathbf{S}_\mathrm{L}^{(r)\mathrm{T}}\mathbf{v}_\mathrm{N}(t), \tag{2}$$

$$\mathbf{v}_\mathrm{R}^{(r)}(t)=\mathbf{S}_\mathrm{R}^{(r)\mathrm{T}}\mathbf{v}_\mathrm{N}(t), \tag{3}$$

where $r = 1,2,...,P$, where m_r is the number of sections of the r-th MTL model, and T denotes a transposition.

The MTLs in Fig. 1 are generally nonuniform, defined by their length l_r and $n_r \times n_r$ p.u.l. matrices $\mathbf{R}_0^{(r)}(x)$, $\mathbf{L}_0^{(r)}(x)$, $\mathbf{G}_0^{(r)}(x)$, and $\mathbf{C}_0^{(r)}(x)$. In their m_r-sectional RLCG models in Fig. 2, the voltage vectors $\mathbf{v}_k^{(r)}$, $k = 1,..., m_r$, and the current vectors $\mathbf{i}_k^{(r)}$, $k = 1,..., m_r + 1$, are the $n_r \times 1$ vectors of the state variables under consideration. The models' lumped parameters are as $\mathbf{R}_k^{(r)} = \mathbf{R}_0^{(r)}(x_k)\Delta x^{(r)}$, $\mathbf{L}_k^{(r)} = \mathbf{L}_0^{(r)}(x_k)\Delta x^{(r)}$, $\mathbf{G}_k^{(r)} = \mathbf{G}_0^{(r)}(x_{k+½})\Delta x^{(r)}$, and $\mathbf{C}_k^{(r)} = \mathbf{C}_0^{(r)}(x_{k+½})\Delta x^{(r)}$, with $x_{k+½} = (x_k + x_{k+1})/2$, and $\Delta x^{(r)} = l_r/m_r$, while $r = 1, 2,..., P$. The border series elements, $\mathbf{R}_1^{(r)}$, $\mathbf{L}_1^{(r)}$, $\mathbf{R}_{m+1}^{(r)}$, $\mathbf{L}_{m+1}^{(r)}$, are taken half-size to ensure the models to be cascade connections of T networks of the same type (identical for uniform MTLs). The terminal currents follow respective state currents as:

$$\mathbf{i}_\mathrm{L}^{(r)}(t)=\mathbf{i}_1^{(r)}(t), \tag{4}$$

$$\mathbf{i}_\mathrm{R}^{(r)}(t)=-\mathbf{i}_{m+1}^{(r)}(t), \tag{5}$$

for all $r = 1,2,...,P$.

As is shown in [15] the hybrid system in Fig. 1 can be described by a system of DAEs:

$$\left[\begin{array}{cccc|c}\mathbf{M}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{(2)} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}^{(P)} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{C}_\mathrm{N}\end{array}\right]\times\frac{d}{dt}\left[\begin{array}{c}\mathbf{x}^{(1)}(t)\\ \mathbf{x}^{(2)}(t)\\ \vdots\\ \mathbf{x}^{(P)}(t)\\ \hline \mathbf{v}_\mathrm{N}(t)\end{array}\right]=$$

$$=\left[\begin{array}{cccc|c}\mathbf{H}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{S}^{(1)\mathrm{T}}\\ \mathbf{0} & \mathbf{H}^{(2)} & \cdots & \mathbf{0} & -\mathbf{S}^{(2)\mathrm{T}}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}^{(P)} & -\mathbf{S}^{(P)\mathrm{T}}\\ \hline \mathbf{S}^{(1)} & \mathbf{S}^{(2)} & \cdots & \mathbf{S}^{(P)} & \mathbf{G}_\mathrm{N}\end{array}\right]\times\left[\begin{array}{c}\mathbf{x}^{(1)}(t)\\ \mathbf{x}^{(2)}(t)\\ \vdots\\ \mathbf{x}^{(P)}(t)\\ \hline \mathbf{v}_\mathrm{N}(t)\end{array}\right]+$$

$$+\left[\begin{array}{c}\mathbf{0}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \hline \mathbf{i}_\mathrm{N}(t)\end{array}\right], \tag{6}$$

with matrices $\mathbf{M}^{(r)}$, $\mathbf{H}^{(r)}$, $\mathbf{S}^{(r)}$, and $\mathbf{x}^{(r)}(t)$, $r = 1,2,..., P$, given by:

$$\mathbf{M}^{(r)}=\mathrm{diag}\left(\mathbf{C}^{(r)},\mathbf{L}^{(r)}\right), \tag{7}$$

$$\mathbf{H}^{(r)}=\begin{bmatrix}\mathbf{G}^{(r)} & \mathbf{E}\\ -\mathbf{E}^\mathrm{T} & \mathbf{R}^{(r)}\end{bmatrix}, \tag{8}$$

where $\mathbf{C}^{(r)} = \mathrm{diag}(\mathbf{C}_1^{(r)},..., \mathbf{C}_m^{(r)})$, $\mathbf{L}^{(r)} = \mathrm{diag}(\mathbf{L}_1^{(r)},..., \mathbf{L}_{m+1}^{(r)})$, $\mathbf{G}^{(r)} = \mathrm{diag}(\mathbf{G}_1^{(r)},..., \mathbf{G}_m^{(r)})$, $\mathbf{R}^{(r)} = \mathrm{diag}(\mathbf{R}_1^{(r)},..., \mathbf{R}_{m+1}^{(r)})$, and $\mathbf{E}$ is a block matrix of the identity and zero matrices, $\pm\mathbf{I}$ and $\mathbf{0}$, respectively, see [15] for more details. A compound selector matrix of the order $N \times n_r(2m_r + 1)$ is formed as

$$\mathbf{S}^{(r)}=[\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m_r\ \text{times}},\mathbf{S}_\mathrm{L}^{(r)},\ \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{(m_r-1)\ \text{times}},\ -\mathbf{S}_\mathrm{R}^{(r)}], \tag{9}$$

containing $N \times n_r$ zero matrices of marked numbers. Finally,

$$\mathbf{x}^{(r)}(t)=\left[\mathbf{v}^{(r)\mathrm{T}}(t),\mathbf{i}^{(r)\mathrm{T}}(t)\right]^\mathrm{T}, \tag{10}$$

is a vector of state variables, $\mathbf{v}^{(r)}(t) = [\mathbf{v}_1^{(r)\mathrm{T}}(t),..., \mathbf{v}_m^{(r)\mathrm{T}}(t)]^\mathrm{T}$, and $\mathbf{i}^{(r)}(t) = [\mathbf{i}_1^{(r)\mathrm{T}}(t),..., \mathbf{i}_{m+1}^{(r)\mathrm{T}}(t)]^\mathrm{T}$, see also Fig. 2, while $\mathbf{v}_\mathrm{N}(t)$ contains all node voltages, appended by chosen currents, of a lumped-parameter part, see Fig. 1. The terminal variables, $\mathbf{v}_\mathrm{L}^{(r)}(t)$, $\mathbf{v}_\mathrm{R}^{(r)}(t)$, and $\mathbf{i}_\mathrm{L}^{(r)}(t)$, $\mathbf{i}_\mathrm{R}^{(r)}(t)$, are determined by (2)–(5), respectively, cp. Fig. 1 and Fig. 2. Further, if we designate:

$$\mathbf{M}=\mathrm{diag}\left(\mathbf{M}^{(1)},\mathbf{M}^{(2)},\ldots,\mathbf{M}^{(P)}\right), \tag{11}$$

$$\mathbf{H}=\mathrm{diag}\left(\mathbf{H}^{(1)},\mathbf{H}^{(2)},\ldots,\mathbf{H}^{(P)}\right), \tag{12}$$

$$\mathbf{S}=\left[\mathbf{S}^{(1)},\mathbf{S}^{(2)},\ldots,\mathbf{S}^{(P)}\right], \tag{13}$$

$$\mathbf{x}(t) = \left[\mathbf{x}^{(1)\mathrm{T}}(t), \mathbf{x}^{(2)\mathrm{T}}(t), \ldots, \mathbf{x}^{(P)\mathrm{T}}(t)\right]^{\mathrm{T}}, \tag{14}$$

the (6) results in

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathrm{N}} \end{bmatrix} \times \frac{d}{dt} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}_{\mathrm{N}}(t) \end{bmatrix} = -\begin{bmatrix} \mathbf{H} & -\mathbf{S}^{\mathrm{T}} \\ \mathbf{S} & \mathbf{G}_{\mathrm{N}} \end{bmatrix} \times \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}_{\mathrm{N}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{i}_{\mathrm{N}}(t) \end{bmatrix}, \tag{15}$$

or, formally written, a resultant DAE can be expressed by

$$\mathbf{A}\frac{d\mathbf{z}(t)}{dt} = \mathbf{B}\mathbf{z}(t) + \mathbf{u}(t), \tag{16}$$

with the system matrices of the order $N + \sum_{r=1}^{P} n_r (2m_r + 1)$.

III. STOCHASTIC EFFECTS CONSIDERATION

We will study a system whose MTLs' p.u.l. conductance or resistance matrices, and consequently models' matrices $\mathbf{G}_k^{(r)}$ or $\mathbf{R}_k^{(r)}$, vary randomly. In such a case, (16) leads to

$$\mathbf{A}\frac{d\mathbf{z}(t)}{dt} = \left(\mathbf{B} + \delta\mathbf{B}(t)\right)\mathbf{z}(t) + \mathbf{u}(t), \tag{17}$$

with $\delta\mathbf{B}(t)$ term expressing stochastic processes, and equal to

$$\delta\mathbf{B}(t) = -\mathrm{diag}\left(\delta\mathbf{H}(t), \mathbf{0}\right), \tag{18}$$

with $N \times N$ zero matrix, as results from (15), and

$$\delta\mathbf{H}(t) = \mathrm{diag}\left(\delta\mathbf{H}^{(1)}(t), \delta\mathbf{H}^{(2)}(t), \ldots, \delta\mathbf{H}^{(P)}(t)\right), \tag{19}$$

as results from (6), or (12), with

$$\delta\mathbf{H}^{(r)}(t) = \mathrm{diag}\left(\delta\mathbf{G}^{(r)}(t), \delta\mathbf{R}^{(r)}(t)\right) = \boldsymbol{\alpha}^{(r)} \circ \boldsymbol{\xi}^{(r)}(t), \tag{20}$$

where $r = 1,2,\ldots, P$, resulting from (8), where $\circ$ marks a Hadamard (entrywise) product of matrices, and:

$$\boldsymbol{\alpha}^{(r)} = \mathrm{diag}\left(\boldsymbol{\alpha}_{\mathrm{G}}^{(r)}, \boldsymbol{\alpha}_{\mathrm{R}}^{(r)}\right), \tag{21}$$

$$\boldsymbol{\alpha}_{\mathrm{G}}^{(r)} = \mathrm{diag}\left(\boldsymbol{\alpha}_{\mathrm{G}1}^{(r)}, \ldots, \boldsymbol{\alpha}_{\mathrm{G}m}^{(r)}\right), \tag{22}$$

$$\boldsymbol{\alpha}_{\mathrm{R}}^{(r)} = \mathrm{diag}\left(\boldsymbol{\alpha}_{\mathrm{R}1}^{(r)}, \ldots, \boldsymbol{\alpha}_{\mathrm{R}(m+1)}^{(r)}\right), \tag{23}$$

denote noise intensities matrices, while:

$$\boldsymbol{\xi}^{(r)}(t) = \mathrm{diag}\left(\boldsymbol{\xi}_{\mathrm{G}}^{(r)}(t), \boldsymbol{\xi}_{\mathrm{R}}^{(r)}(t)\right), \tag{24}$$

$$\boldsymbol{\xi}_{\mathrm{G}}^{(r)}(t) = \mathrm{diag}\left(\boldsymbol{\xi}_{\mathrm{G}1}^{(r)}(t), \ldots, \boldsymbol{\xi}_{\mathrm{G}m}^{(r)}(t)\right), \tag{25}$$

$$\boldsymbol{\xi}_{\mathrm{R}}^{(r)} = \mathrm{diag}\left(\boldsymbol{\xi}_{\mathrm{R}1}^{(r)}(t), \ldots, \boldsymbol{\xi}_{\mathrm{R}(m+1)}^{(r)}(t)\right), \tag{26}$$

denote "white noise processes" matrices. The above stated matrices (18)–(26) are symmetric due to symmetries of all the MTLs' p.u.l. matrices [15].

Matching to Fig. 2, non-deterministic shunt conductances and series resistances can respectively be expressed as

$$\mathbf{G}_k^{*(r)}(t) = \mathbf{G}_k^{(r)} + \boldsymbol{\alpha}_{\mathrm{G}k}^{(r)} \circ \boldsymbol{\xi}_{\mathrm{G}k}^{(r)}(t), \tag{27}$$

where $k = 1,\ldots, m_r$, and

$$\mathbf{R}_k^{*(r)}(t) = \mathbf{R}_k^{(r)} + \boldsymbol{\alpha}_{\mathrm{R}k}^{(r)} \circ \boldsymbol{\xi}_{\mathrm{R}k}^{(r)}(t), \tag{28}$$

where $k = 1,\ldots, m_r + 1$, $r = 1, 2,\ldots, P$, with the symmetrical matrices $\boldsymbol{\alpha}_{\mathrm{G}k}^{(r)} = [\alpha_{\mathrm{G}ij}]_k^{(r)}$, $\boldsymbol{\xi}_{\mathrm{G}k}^{(r)}(t) = [\xi_{\mathrm{G}ij}(t)]_k^{(r)}$, and $\boldsymbol{\alpha}_{\mathrm{R}k}^{(r)} = [\alpha_{\mathrm{R}ij}]_k^{(r)}$, $\boldsymbol{\xi}_{\mathrm{R}k}^{(r)}(t) = [\xi_{\mathrm{R}ij}(t)]_k^{(r)}$, $i,j = 1,\ldots, n_r$, with equal (i,j)-th and (j,i)-th elements, for $i \neq j$.

A conversion of DAE (17) into its stochastic counterpart consists in multiplying it by dt and then replacement of all the products $\xi_{ij,k}^{(r)}(t)dt$ by $dW_{ij,k}^{(r)}(t)$, with $W_{ij,k}^{(r)}(t)$ denoting Wiener processes [1]. By this we get a linear stochastic DAE (SDAE) with multiplicative noises

$$\mathbf{A}d\mathbf{Z}(t) = \left(\mathbf{B}\mathbf{Z}(t) + \mathbf{u}(t)\right)dt + \delta\mathbf{B}_{\mathrm{W}}(t)\mathbf{Z}(t), \tag{29}$$

where $\delta\mathbf{B}_{\mathrm{W}}(t)$ arises from (18), after making the above stated operations and introducing respective Wiener processes. It is common to mark a stochastic solution by a capital letter, $\mathbf{Z}(t)$, to distinguish it from a deterministic solution, $\mathbf{z}(t)$.

IV. SDAE NUMERICAL SOLUTION

As is obvious from (15) the matrix $\mathbf{A}$ in (16), (17), and thus in (29), is singular in general due to a singularity of $\mathbf{C}_{\mathrm{N}}$ matrix in (1), which leads to discussed DAEs and SDAEs. There are various approaches for the solution of the SDAEs, see e.g. [4], [5], [16], depending mainly on their index. For example, it is first possible to decouple the SDAE into its differential and algebraic parts by using a canonical projectors method [5]. The former can be regarded as the SDE describing Itô stochastic processes, the other as its algebraic constraints. However, this procedure is not recommended for the numerical solution due to possible numerical instabilities [5]. Instead, in practice, common one- or multi-step backward differentiation formulae are usually applied for the index-1 SDAEs [16].

Let us consider solutions $\mathbf{Z}(t)$ on the interval $t_0 \leq t \leq T$, in conjunction with an equidistant division $t_l = t_0 + lh$, with $h = (T - t_0)/N_T = t_{l+1} - t_l$, for $l = 0,\ldots, N_T-1$, and with the relating processes $\Delta W^{(l)} = W(t_{l+1}) - W(t_l) = \int_{t_l}^{t_{l+1}} dW(\tau)$. For numerical simulations we have to generate the random increments of the Wiener processes as independent Gauss random variables, with means $E[\Delta W^{(l)}] = 0$ and $E[(\Delta W^{(l)})^2] = h$. Then, a stochastic backward Euler scheme can be written as

$$\mathbf{Z}^{(l+1)} = \left(\mathbf{A} + \mathbf{B}h\right)^{-1} \cdot \left(\mathbf{u}^{(l+1)}h + \left(\mathbf{A} + \delta\mathbf{B}_{\mathrm{W}}^{(l)}\right)\mathbf{Z}^{(l)}\right), \tag{30}$$

which is consistent with the Itô calculus. The matrix $\mathbf{A} + \mathbf{B}h$ is now regular as $\mathbf{C}_{\mathrm{N}} + \mathbf{G}_{\mathrm{N}}$ is now a regular matrix.

Our problem, however, does not require a good pathwise approximation but only an approximation of the expectation $\mu = E[Z]$ and the standard deviation $\sigma = \sqrt{E[(Z-\mu)^2]}$ of the SDAE solutions. We can use the weak version of (30), where we replace the increments of the Wiener processes $\Delta W^{(l)}$ by simplier two-point distributed random variables $\Delta\widehat{W}^{(l)}$, where $P(\Delta\widehat{W}^{(l)} = \pm\sqrt{h}) = ½$. A numerical efficiency of the weak scheme is better than of the strong one. That is important in our problem of MTL systems, where we have a large system with many independent Wiener processes.

V. SIMULATION EXAMPLE

A hybrid system with three MTLs, each with two active conductors, $n_r = 2$, $r = 1, 2, 3$, is shown in Fig. 3. The MTLs have lengths: $l_1 = 0.05$ m, $l_2 = 0.04$ m, and $l_3 = 0.03$ m [18].

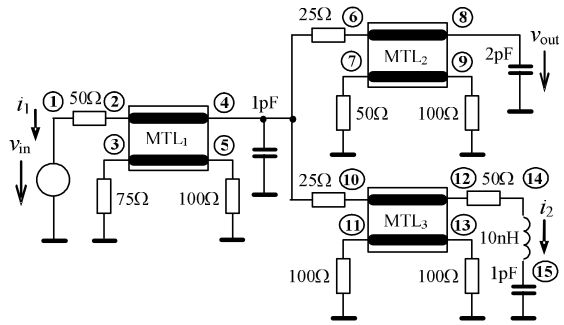
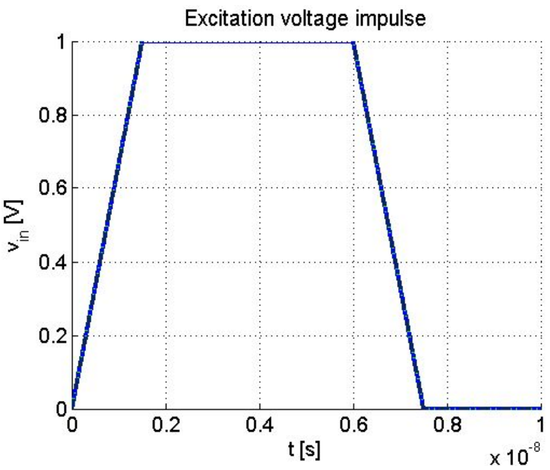
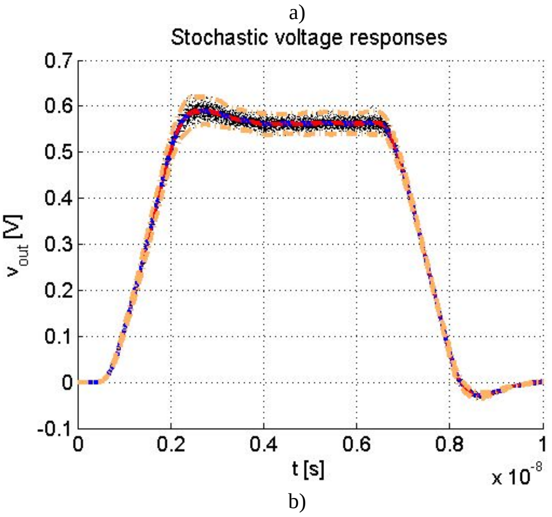


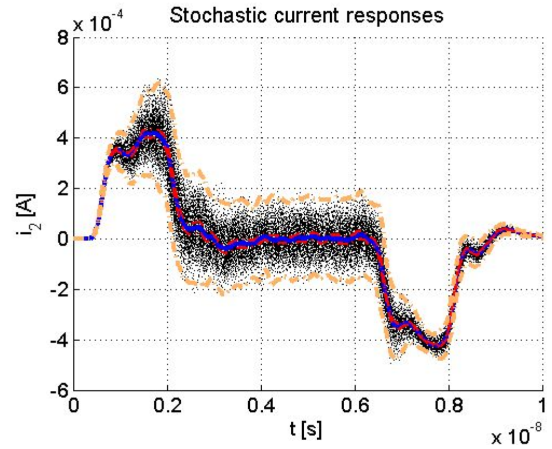
Fig. 3. Hybrid system with fluctuating MTLs parameters.



a)



b)



c)

Fig. 4. Stochastic responses with mean values and confidence intervals: a) excitation impulse v_{in}; b) output voltage v_{out}; c) inductor current i_2.

A 1 V pulse with 1.5 ns rise/fall time and 7.5 ns width is at the input, $v_{in}(t)$, the MTLs models' resistances $R_k^{(r)}(i,i)$, $i = 1,2$, $\forall r$, $k = 1,..., 128$, are set fluctuating, with the noise intensities $\alpha_{Rii}^{(r)} = 5\cdot10^{-6}$, while total resistances of the wires result from MTLs p.u.l. resistance 75 Ω/m and lengths [18]. The stochastic responses at the MTL$_2$ output, $v_{out}(t)$, including excitation voltage $v_{in}(t)$, and the MTL$_3$ output, $i_2(t)$, are shown in Fig. 4. Sample statistics were stated through 100 realizations, 99 % confidence intervals (dash red) for mean values (solid blue), and individual samples (dash orange, for vicinities circa ±2.6 σ), were obtained via a student-*t* distribution [2]. More detailed information about the statistical estimates is shown e.g. in [2], [9], [17]. The sample means were also verified via deterministic solutions.

VI. CONCLUSIONS

The paper has dealt with a method for the simulation of stochastic responses in hybrid systems containing MTLs with randomly varying parameters based on SDEs theory. The method was developed for fluctuating series resistances and shunt conductances of respective MTLs' models. The work follows concepts in [14], [15], when by using the MNA, the MTLs can be integrated into arbitrarily complex hybrid systems. By this, however, a system of stochastic DAEs is generally obtained which is usually more difficult to solve. By contrast to the previous works, for numerical simulations we used a weak backward Euler scheme that is more effective to solve sample moments. Besides, the higher order stochastic numerical schemes, we plan to use in future, involve multiple Itô integrals of higher multiplicity which are difficult to generate. In weak schemes these multiple integrals can be replaced by simpler random variables and we can obtain quite efficient higher order weak schemes.

All simulations were performed in the Matlab language utilizing techniques of sparse matrices processings. The above stated approach could be an interesting alternative to other stochastic or probabilistic methods, like e.g. [19]–[21] often used to study random changes in interconnects of high-speed electronic systems. Our work is focused on improving the efficiency of the numerical simulations in this field.

REFERENCES

[1] B. Oksendal, *Stochastic Differential Equations, An Introduction with Applications*. New York: Springer-Verlag, 2000.

[2] P. Kloeden, E. Platen, H. Schurz, *Numerical Solution of SDE Through Computer Experiments*. New York: Springer-Verlag, 1997.

[3] D. Ham, A. Hajimiri, "Complete noise analysis for CMOS switching mixers via stochastic differential equations", in *Proc. of IEEE Custom Integrated Circuits Conf.*, Orlando, 2000, pp. 439–442.

[4] K. Wang, M. L. Crow, "Numerical simulation of stochastic differential algebraic equations for power system transient stability with random loads", in *Proc. 2011 IEEE Power and Energy Society General Meeting*, San Diego, 2011, pp. 1–8.

[5] O. Schein, G. Denk, "Numerical solution of stochastic differential algebraic equations with applications to transient noise simulation of microelectronic circuits", *Journal of Computational and Applied Mathematics*, vol. 100, no. 1, 1998, pp. 77–92. [Online]. Available: http://dx.doi.org/10.1016/S0377-0427(98)00138-1

[6] C. K. Cheng, J. Lillis, S. Lin, N. Chang, *Interconnect Analysis and synthesis*. New York: John Wiley & Sons, 2000.

[7] C. R. Paul, *Analysis of Multiconductor Transmission Lines*. New York: John Wiley & Sons, 2008.

[8] M. P. LI, *Jitter, Noise, and Signal Integrity at High-Speed*. Upper Saddle River: Prentice Hall, 2007.

[9] E. Kolarova, "Statistical estimates of stochastic solutions of RL electrical circuits", in *Proc. the IEEE Int. Conf. Industrial Technology (ICIT 2006)*, Mumbai, 2006, pp. 2546–2550. [Online]. Available: http://dx.doi.org/10.1109/ICIT.2006.372644

[10] E. Kolarova, L. Brancik, "Application of stochastic differential equations in second-order electrical circuits analysis", *Przeglad Elektrotechniczny (Eectrical Review)*, vol. 88, no. 7a, pp. 103–107, 2012.

[11] L. Brancik, E. Kolarova, A. Prokes, "Simulation of random effects in transmission line models via stochastic differential equations", in *Proc. the 2nd Int. Conf. Advances in Computational Tools for Engineering Applications (ACTEA 2012)*, Zouk Mosbeh, Lebanon, Notre Dame University, 2012, pp. 308–312.

[12] L. Brancik, E. Kolarova, "Simulation of higher-order electrical circuits with stochastic parameters via SDEs", *Advances in Electrical and Computer Engineering*, vol. 13, no. 1, pp. 17–22, 2013. [Online]. Available: http://dx.doi.org/10.4316/aece.2013.01003

[13] L. Brancik, E. Kolarova, "Stochastic differential equations approach in the analysis of MTLs with randomly varied parameters", in *Proc. 19th IEEE Int. Conf. Electronics, Circuits, and Systems (ICECS 2012)*, Sevilla, 2012, pp. 725–728.

[14] L. Brancik, E. Kolarova, "Simulation of stochastic responses at multiconductor transmission lines with fluctuating parameters", in *Proc. 23th Int. Conf. Radioelektronika*, Pardubice, Czech Rep., University of Pardubice, 2013, pp. 61–64.

[15] L. Brancik, "Simulation of hybrid MTL systems with random parameters based on stochastic DAEs", in Proc. *21st European Conf. Circuit Theory and Design (ECCTD 2013)*, Dresden, Germany, 2013, pp. 1–4. [Online]. Available: http://dx.doi.org/10.1109/ECCTD.2013.6662336

[16] R. Winkler, "Stochastic differential algebraic equations in transient noise analysis", in *Scientific Computing in Electrical Engineering*, vol. 9, pp. 151–156, 2006.

[17] S. Cyganowski, P. Kloeden, J. Ombach, *From Elementary Probability to Stochastic Differential Equations with MAPLE*. New York: Springer-Verlag, 2002. [Online]. Available: http://dx.doi.org/10.1007/978-3-642-56144-3

[18] S. Lum, M. S. Nakhla, Q. J. Zhang, "Sensitivity analysis of lossy coupled transmission lines", *IEEE Trans. Microwave Theory and Techniques*, vol. 39, no. 12, pp. 2089–2099, 1991. [Online]. Available: http://dx.doi.org/10.1109/22.106551

[19] D. Spina, F. Ferranti, T. Dhaene, L. Knockaert, G. Antonini, D. V. Ginste, "Variability analysis of multiport systems via polynomial-chaos expansion", *IEEE Trans. Microwave Theory and Techniques*, vol. 60, no. 8, pp. 2329–2338, 2012. [Online]. Available: http://dx.doi.org/10.1109/TMTT.2012.2202685

[20] D. V. Ginste, D. De Zutter, D. Deschrijver, T. Dhaene, P. Manfredi, F. Canavero, "Stochastic modeling-based variability analysis of on-chip interconnects", *IEEE Trans. Components, Packaging and Manufacturing Technology*, vol. 2, no. 7, pp. 1182–1192, 2012. [Online]. Available: http://dx.doi.org/10.1109/TCPMT.2012.2192274

[21] D. Eidukas, "Quality Level Linear Models Electronic Systems", *Elektronika ir Elektrotechnika*, no. 3, pp. 57–60, 2012. [Online]. Available: http://dx.doi.org/10.5755/j01.eee.119.3.1364