http://dx.doi.org/10.5755/j01.eee.19.10.5911 ELEKTRONIKA IR ELEKTROTECHNIKA, ISSN 1392-1215, VOL. 19, NO. 10, 2013

Design of Compandor Based on Approximate the First-Degree Spline Function

L. Velimirovic[1], Z. Peric[2], M. Stankovic[3], J. Nikolic[2]
[1]*Mathematical Institute of the Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11001 Belgrade, Serbia*
[2]*Faculty of Electronic Engineering Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia*
[3]*Faculty of Occupational Safety Niš, Čarnojevića 10 A, 18000 Niš, Serbia*
velimirovic.lazar@gmail.com

***Abstract*—In this paper, the approximation of the optimal compressor function using spline function of the first-degree is done. For the companding quantizer designed on the basis of the approximative spline function of the first-degree, the support region is numerically optimized to provide the minimum of the total distortion for the last segment. It is shown that the companding quantizer with the optimized support region threshold provides the signal to quantization noise ratio that is very close to the one of the optimal companding quantizer having an equal number of levels.**

***Index Terms*—Optimization methods, quantization, spline.**

I. INTRODUCTION

In digital signal processing, quantization is the process of approximating a continuous range of values (or a very large set of possible discrete values) by a relatively-small set of discrete values. A device that performs quantization is called a quantizer [1]. Quantizers can be uniform and nonuniform. It is well known that uniform quantizers are suitable for signals that have approximately uniform probability density function (PDF) [1]. Since many of the real signals are characterized by continuous Laplacian random variable, which means that some input values are expected to be more often than the others, nonuniform quantization is more common case in practice. Nonuniform quantization can be realized through the process of companding, in which a specific compressor function is applied on an input signal [1]. Specifically, nonuniform quantization can be achieved by compressing the signal x using a nonuniform compressor characteristic $c(\cdot)$, by quantizing the compressed signal $c(x)$ employing a uniform quantizer, and by expanding the quantized version of the compressed signal using a nonuniform transfer characteristic $c^{-1}(\cdot)$ that is inverse to that of the compressor. The overall structure of a nonuniform quantizer consisting of a compressor, a uniform quantizer, and an expandor in cascade is called compandor or companding quantizer [1]. Although the optimal compressor function gives the maximum of the signal to quantization noise ratio (SQNR) at the reference variance at which the optimal companding quantizer is designed, the optimal companding quantizer is not widely used because it is very complicated to be realized practically [1]. In order to provide easier practical realization, approximation of the optimal compressor function is performed. Accordingly, in this paper, the approximation of the optimal compressor function using spline function of the first-degree, for Laplacian PDF is done. The primary goal of the companding quantizer design is to achieve as high as possible the SQNR for as small as possible complexity [2].

A quantizer support region can be divided into a variety of ways. Unlike with the quantizers described in [3]–[6], the support region of the proposed quantizer model is not divided into segments of equal width. In [6], the approximation of the optimal compressor function using spline function of the first and second degree, for Laplacian PDF is done. As already mentioned, the support region of the quantizer described in [6] is divided into equal in width segments, each of which has an unequal number of cells. This result in a higher complexity of encoding and decoding procedure compared to the case where the number of cells per segments are equal. In order to overcome the mentioned problems with the realization of the optimal companding quantizer, and also to decrease the encoding and decoding complexity, in this paper we develop a new method of construction companding quantizers which introduces an equal number of cells within segments of unequal size. Also, in order to improve the performance of our companding quantizer, the method we propose introduces the optimization of the support region threshold. By designing the proposed quantizer based on the approximate spline function and with optimized support region threshold, SQNR that is close to that of the nonlinear optimal companding quantizer is obtained.

The rest of the paper is organized as follows: In Section II the detailed description of spline functions of the first-degree and optimal compressor function is given. Design of the quantizer based on the approximate spline function of the

Manuscript received January 16, 2013; accepted June 8, 2013.
This work is partially supported by Serbian Ministry of Education and Science through Mathematical Institute of Serbian Academy of Sciences and Arts (Project III44006) and by Serbian Ministry of Education and Science (Project TR32035).

first-degree is described in Section III. Also, the procedure for optimization of the optimal support region threshold is described in Section III. Finally, the achieved numerical results for the Laplacian source of unit variance are discussed in Section IV.

II. Approximations Using Spline Functions of the First-Degree

In this paper, the approximation of the optimal compressor function using the spline function of the first-degree is done. The spline function is a function that consists of polynomial pieces joined together with certain smoothness conditions [7]. A simple example is the polygonal function (or spline of degree 1), whose pieces are linear polynomials joined together to achieve continuity [7]. In the theory of splines, the points $x_0, x_1,..., x_L$ at which the function changes its character are termed knots. Such a function appears somewhat complicated when defined in explicit terms. Accordingly, we consider the following definition of a linear polynomial $S(x)$ [7]:

$$S(x)=\begin{cases} S_0(x), & x\in[x_0,x_1], \\ S_1(x), & x\in[x_1,x_2], \\ \vdots & \\ S_{L-1}(x), & x\in[x_{L-1},x_L], \end{cases} \tag{1}$$

where

$$S_i(x)=a_i x+b_i. \tag{2}$$

Obviously, $S(x)$ is a piecewise linear function. For the given knots $x_0, x_1,..., x_L$ and coefficients $a_0, b_0, a_1, b_1,..., a_{L-1}, b_{L-1}$, the evaluation of $S(x)$ at a specific x performs by first determining the interval that contains x and then by using the appropriate linear function for that interval. If the function S defined by (1) is continuous, we call it a first-degree spline. It is characterized by the following three properties [7].

Definition 1. A function S is called a spline of the first-degree if:

1. The domain of S is an interval $[a, b]$;
2. S is continuous on $[a, b]$;
3. There is a partitioning of the interval $a = x_0 < x_1 <_{...} < x_L = b$ such that S is a linear polynomial on each subinterval $[x_i, x_{i+1}]$.

Outside the interval $[a, b]$, $S(x)$ is usually defined to be the same function on the left of a as it is on the leftmost subinterval $[x_0, x_1]$ and the same on the right of b as it is on the rightmost subinterval $[x_{L-1}, x_L]$ [7]. In other words, $S(x) = S_0(x)$ when $x < a$ and $S(x) = S_{L-1}(x)$ when $x > b$. Continuity of a function f at a point s can be defined by the condition [7]

$$\lim_{x\to s^+} f(x)=\lim_{x\to s^-} f(x)=f(s), \tag{3}$$

where, $\lim_{x\to s^+}$ means that the limit is taken over x values that converge to s from above s; that is, $(x-s)$ is positive for all x values. Similarly, $\lim_{x\to s^-}$ means that the x values converge to s from below.

The first-degree spline, also called the polygonal function, is consisted of line segments that are connected so that given function is continuous. As alredy mentioned, the points where the function changes its shape are called knotes [7]. The approximate function $g(x)$, by which a nonlinear compressor function $c(x)$ is approximated in this paper, for the number of segments L, has the following form [7]

$$g(x)=\begin{cases} c(x_1)+m_1(x-x_1), & x\in[0,x_1], \\ c(x_i)+m_i(x-x_i), & x\in[x_{i-1},x_i], \quad i=2,\dots,L, \end{cases} \tag{4}$$

where m_i is the coefficient of direction of the line given by [7]

$$m_i=\frac{c(x_i)-c(x_{i-1})}{x_i-x_{i-1}}, \tag{5}$$

where $i=1,\dots,L$. The optimal compressor function $c(x)$: $[-x_{\max}, x_{\max}] \rightarrow [-x_{\max}, x_{\max}]$ by which the maximum SQNR is achieved for the reference variance of an input signal having PDF $p(x)$ is defined as [1]

$$c(x)=\begin{cases} x_{\max}\dfrac{\int_0^x p^{1/3}(x)dx}{\int_0^{x_{\max}} p^{1/3}(x)dx}, & 0\le x\le x_{\max}, \\ -x_{\max}\dfrac{\int_x^0 p^{1/3}(x)dx}{\int_{-x_{\max}}^0 p^{1/3}(x)dx}, & -x_{\max}\le x\le 0. \end{cases} \tag{6}$$

Without diminishing the generality, in what follows the quantizer design will be done for the reference input variance of $\sigma_{ref}^2=1$.

III. Design of Compandor Based on Approximate the First-Degree Spline Function

This section provides us with a detailed description of the scalar compandor designed according the approximative spline function of the first degree. The support region threshold of the N-level companding quantizer (the granular region threshold) is defined as follows [8]

$$x_{\max}=\frac{3}{\sqrt{2}}\ln\left(\frac{N+1}{3}\right). \tag{7}$$

Let us assume, as in [3], that in the granular region $[-x_{\max}, x_{\max}]$, the total number of the reproduction levels per segments in the first quadrant is

$$\sum_{i=1}^{L}\frac{N_i}{2}=\frac{N-2}{2}, \tag{8}$$

where the number of reproduction levels per segments, $N_i/2$, is determined from the following condition

$$\frac{N_i}{2} = \frac{N}{2L}, \tag{9}$$

where $i = 1,\ldots,L-1$. Then, the number of reproduction levels in the last segment belonging to the granular region is $N_L/2$

$$\frac{N_L}{2} = \frac{N-2}{2} - \left(L-1\right)\frac{N}{2L} = \frac{N}{2L} - 1 \tag{10}$$

and there is only two symmetric reproduction levels within the region complement to the granular region, called the overload region. In other words, as in [4], we assume equal number of reproduction levels per segments.

The segment thresholds of our companding quantizer are determined as follows [1]

$$x_i = c_i^{-1}\left(i\frac{N}{2L}\right), \tag{11}$$

where $i = 1,\ldots,L-1$ and it obviously holds $x_L = x_{\max}$. Cells lengths per segments of the considered companding quantizer are given by

$$\Delta_{i,j} = \frac{\Delta}{g_i^{'}\left(x\right)}, \tag{12}$$

where $i = 1,\ldots,L,\ j = 1,\ldots,\frac{N_i}{2}$ and

$$\Delta = \frac{2x_{\max}}{N-2}. \tag{13}$$

Recall that $g(x)$ is the approximate function, by which we approximate the nonlinear optimal compressor function $c(x)$ (6). Denoted by $\Delta_{i,j}$ are the j-th cells lengths within the i-th segment. The cells thresholds of the considered quantizer are determined as follows

$$x_{i,j}^{\text{cell}} = \frac{j\Delta}{g_i^{'}\left(x\right)}, \tag{14}$$

where $i = 1,\ j = 1,\ldots,\frac{N_i}{2}$.

$$x_{i,j}^{\text{cell}} = \frac{c_i\left(x_i\right) + j\Delta}{g_i^{'}\left(x\right)}, \tag{15}$$

where $i = 2,\ldots,L,\ j = 1,\ldots,\frac{N_i}{2}$.

The granular distortion for the proposed quantizer model is defined by [1]

$$D_g = 2\sum_{i=1}^{L}\sum_{j=1}^{\frac{N_i}{2}}\frac{\Delta_{i,j}^2}{12}P_{i,j}, \tag{16}$$

where $i = 1,\ldots,L,\ j = 1,\ldots,\frac{N_i}{2}$, $P_{i,j}$ denotes the probability of the input sample x of variance σ^2 belonging to the j-th cells in i-th segment. For the assumed Laplacian PDF of unit variance [1]

$$p(x) = \frac{1}{\sqrt{2}\sigma}e^{-\frac{|x|\sqrt{2}}{\sigma}}, \tag{17}$$

we can derive the following closed-form expressions for the probablities $P_{i,j}$

$$P_{i,j} = \int_{x_{i,j-1}^{\text{cell}}}^{x_{i,j}^{\text{cell}}} p\left(x\right)dx = \frac{1}{2}\left[\exp(-\sqrt{2}x_{i,j-1}^{\text{cell}}) - \exp(-\sqrt{2}\,x_{i,j}^{\text{cell}})\right]. \tag{18}$$

The overload distortion D_{o} is defined by [1]

$$D_o = 2\int_{x_{\max}}^{\infty}\left(x - y_N\right)^2 p\left(x\right)dx = \frac{1}{2}\exp\left(-\sqrt{2}x_{\max}\right), \tag{19}$$

where y_N is, as in [3] and [9], determined from the centroid condition

$$y_N = \frac{\int_{x_{\max}}^{\infty} xp\left(x\right)dx}{\int_{x_{\max}}^{\infty} p\left(x\right)dx}. \tag{20}$$

The quality of a quantizer can be measured by distortion or more conveniently by signal to quantization noise ratio (SQNR) [1]. By determining the total distortion D as a sum of the granular distortion D_g (16) and the overload distortion D_{o} (19), the SQNR can be determined [1]

$$\text{SQNR} = 10\log\left(\frac{\sigma^2}{D_g + D_o}\right) = 10\log\left(\frac{\sigma^2}{D}\right). \tag{21}$$

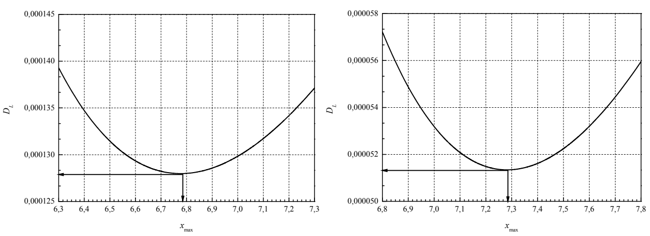
In order to improve the performance of the proposed companding quantizer, we propose a new method of construction the companding quantizer. Our method introduces the optimization of the support region threshold, i.e. the optimization of the approximate spline of degree 1 in the last segment. Numerical determination of the optimal support region threshold is performed respecting the criterion of minimum distortion D_L for the last segment

$$D_L = 2\frac{\left(\dfrac{x_{\max}^{\text{opt}} - x_{L-1}}{N/2L-1}\right)^2}{12}\int_{x_{L-1}}^{x_{\max}^{\text{opt}}} p\left(x\right)dx + \frac{1}{2}\exp\left(-\sqrt{2}x_{\max}^{\text{opt}}\right). \tag{22}$$

In other words, we perform optimization of D_L with respect to the support region threshold. For the optimized support region threshold x_{max}^{opt}, obtained in this way, we design approximate spline of degree 1 and we determine other parameters required to design the described companding quantizer. Numerical results that follow show that by designing the proposed companding quantizer based on the approximate spline of degree 1 and with the optimized support region threshold, SQNR that is close to that of the nonlinear optimal companding quantizer is obtained.

IV. NUMERICAL RESULTS

Numerical results presented in this section are obtained for the case when the number of levels is equal to $N = 128$ and for the number of segments $2L = 8$ and $2L = 16$. In Fig. 1 and Fig. 2 the dependence of the distortion D_L for the last segment of the proposed companding quantizer on the support region threshold x_{max} for the number of segments $2L = 8$ and $2L = 16$ is shown. Based on Fig. 1 it can be concluded that the minimum value of the total distortion D_L of the proposed companding quantizer is achieved for the optimal value of the support region threshold $x_{max}^{opt} = 6.78$. Also, based on Fig. 2 it can be concluded that the minimum value of the total distortion D_L of the proposed companding quantizer is achieved for the optimal value of the support region threshold $x_{max}^{opt} = 7.28$.



(a) (b)

Fig. 1. Numerical determination of the optimal support region threshold for the number of segments: a – $2L = 8$; b – $2L = 16$.

Table I shows the values of SQNR for the proposed companding quantizer and the case where the support region threshold is determined by (7), ($SQNR^F$), the values of SQNR for the proposed companding quantizer and the case where the support region threshold x_{max}^{opt} is numerically optimized, ($SQNR^N$), and the values of SQNR of the optimal companding quantizer having $c(x)$ given by (6), ($SQNR^O$).

TABLE I. THE VALUES OF $SQNR^F$, $SQNR^N$ AND $SQNR^O$ FOR THE NUMBER OF SEGMENTS $2L = 8$ AND $2L = 16$ AND $N = 128$.

2L	$SQNR^F$ [dB]	$SQNR^N$ [dB]	$SQNR^O$ [dB]
8	34.19	34.76	35.71
16	35.21	35.41	35.71

The number of segments assumed is $2L = 8$ and $2L = 16$. Analyzing the results shown in Table I one can conclude that design of the proposed companding quantizer based on the approximate spline of degree 1, with the optimized support region x_{max}^{opt}, provides SQNR very close to that of the optimal companding quantizer having the same number of levels. In addition, given fixed number of levels $N = 128$, $2L = 8$ and $2L = 16$, numerical comparison reveals that the gain in SQNR achieved with the proposed quantizer model compared to the one proposed in [5] ranges up to 0.12 dB.

Since the model proposed in [5] assumed an unequal number of reproduction levels per segments, the encoding and decoding procedure are rather complex than the one we propose. Recall that in this paper we have assumed an equal number of reproduction levels per segments, which is an additional benefit of the proposed model in comparison to the model reported in [5]. Eventually, by comparing the performance (SQNR) of the proposed companding quantizer with the quantizer model having equal number of reproduction levels per segments and the segment thresholds determined in accordance with the equidistant partition of the optimal compressor function [4], we have reveled a higher quality of the quantized signal, that is a higher SQNR for about 0.85 dB, for the same total number of levels $N = 128$ and for the number of segments $2L = 16$.

V. CONCLUSIONS

In this paper, we have presented and analyzed the companding quantizer with the optimized support region threshold designed based on the approximative spline function of the first degree. From obtained results and comparison to other models, it can be concluded that the proposed model represents a significantly better solution. Also, since the model proposed in [4], designed for the same PDF, have found application in coding of speech signals, we expect that the model proposed in this paper will find a similar application. This will be the subject of our further research.

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