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The Influence of Probability Density Function Discretization on Geometric Lattice Quantizer Design for Memoryless Gaussian Source

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***Abstract*—In this paper the geometric cubic lattice quantizer design based on the PDF discretization is analyzed for the two-dimensional Gaussian source. Particularly, inside the regions, obtained by the geometric support partition, the constant PDF of the input signal vector is supposed. For this input vector PDF approximation and for the given quantizer rate, the granular distortion is optimized in order to get the manner of total points number distribution per regions. Also, the expression for the granular distortion is determined and used to estimate the performance of the proposed model. The SQNR of the proposed quantizer is compared with the known optimal ratio and on these bases it is concluded, among the other things, under which condition the suggested approximation can be applied.**

***Index Terms*—Polar quantization, probability density function discretization, piecewise uniform lattice quantization.**

I. INTRODUCTION

It is known that uniform quantizer is optimal for uniform distribution, that is piecewise uniform quantizer is optimal for piecewise uniform probability density function (PDF) [1], [2]. This fact implies that a near optimum vector quantizer can be designed by finding an optimum quantizer for an adequate PDF discretization. Particularly, as the approximation of a nonuniform with a piecewise uniform distribution becomes more accurate, the asymptotic distortion approaches its minimum value.

The piecewise uniform vector quantizer design based on input vector PDF discretization doesn't appear frequently in the literature. Usually, the piecewise uniform vector quantizer design is based on a compression function linearization [3]. The idea of vector quantizer design based on the input vector PDF discretization has been firstly presented in [2], [4]. The authors of [2] have cited in the conclusion that the main shortage of their analysis is lack of a method for deciding how to change the regions in which to divide the input space. In an example of a two-dimensional Gaussian distribution, the support has been chosen to be a hexagon, while regions have been heuristically selected to be trapezoids. The primary goal of the quantizer design is to determine the support partition such as to provide the minimum possible distortion for the considered number of quantization cells N, or equivalently for the quantizer rate $R = \log_2 N/2$ [5].

In general, the support partition of polar quantizers [4] are defined by the Gaussian source geometry. In [4], the optimal piecewise uniform quantizer for that source geometry has been designed assuming four cases for the piecewise uniform approximation of the input radius PDF. However, the lack of analysis conducted in [4] is that an approximate expression for the distortion calculation has been used.

In this paper we consider two-dimensional cubic lattice quantizer of memoryless Gaussian source and we examine possibilities of its designing under the assumption that the smooth curve of the input vector PDF is the piecewise uniform characteristic. In order to provide better following of the statistical characteristics of signal, we apply a geometric approach in the quantizer designing [6]. This means that the support region is partitioned into regions having boundaries as contours of constant input vector PDF. In such a manner the cell length is better adapted to the source PDF. For the two-dimensional distribution of independent Gaussian variables, we obtain that the support represents circle and the regions are concentric rings. Besides, in order to achieve low implementation complexity, we apply highly structured cubic lattice quantization [1] inside the regions. We perform asymptotic analysis applying the piecewise uniform approximation for the input vector PDF, which was obtained similarly as in [4]. We find the lattice cell length for each region on the base of minimum distortion criterion. Indeed, by optimizing the granular distortion of the considered model, we determine the closed form expression for the number of points inside each region. Usefulness of quantizer designing based on the PDF discretization is estimated on the base of the comparison between the obtained distortion and the known optimal distortion [7]. We emphasize that quantizer is designed using asymptotic analysis that is simplified by assuming the PDF approximation. Moreover, the more accurate formula for the distortion of the proposed quantizer is derived and used for the performance estimation.

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II. Quantizer Model based on PDF Discretization

We consider a geometric piecewise uniform two-dimensional lattice quantization of memoryless Gaussian source. Without the loss of generality, we suppose that the input signal samples x_i are independent Gaussian variables with zero mean value and variance 1, which means that the joint probability density function of samples x_i, $i = 1,2$ is [1], [4]

$$f(\mathbf{x}) = \frac{1}{2\pi} \cdot \exp\left(-\frac{x_1^2 + x_2^2}{2} \right). \tag{1}$$

Then, the contour of constant probability density function is given by [4]

$$x_1^2 + x_2^2 = -2\ln\left(2\pi f_c\right), \tag{2}$$

where f_c is the value of probability density function. Equation (2) is an expression for circle, whose radius [4]

$$r = \sqrt{x_1^2 + x_2^2} \tag{3}$$

has value $-\ln(2\pi f_c)$. This contour of constant PDF is of great significance for the geometric piecewise uniform vector quantizer design since it represents the region boundary. Namely, the ith region R_i is a ring having boundaries as circles with radii r_{i-1} and r_i. Region R_i defined in such a manner has surface S_i

$$S_i = \left(r_i^2 - r_{i-1}^2\right)\pi, \quad i = 1,\ldots,N_q, \tag{4}$$

where N_q is the number of regions. The presented space partition into regions is the first step in the geometric piecewise uniform lattice quantizer designing, while the lattice grid application inside obtained regions is the second step. Thus, besides boundary radii r_i, $i = 0,\ldots, N_q$ and region surfaces S_i, $i = 1,\ldots, N_q$, we consider the cell number and the cell side length inside regions N_i, $i = 1,\ldots,N_q$ and Δ_i, $i = 1,\ldots,N_q$, respectively. Taking into consideration that analysis is asymptotic, relation between Δ_i and N_i can be written as

$$\Delta_i = \left(\frac{S_i}{N_i}\right)^{\frac{1}{2}}, \quad i = 1,\ldots,N_q. \tag{5}$$

Equations (5) and (4) show that for the given geometric space partition into regions, the quantizer designing can be completed if N_i is determined.

Finally, the granular distortion per dimension for the piecewise uniform cubic lattice quantizer can be written as [1]

$$D_g = \frac{1}{12}\sum_{i=1}^{N_q} \Delta_i^2 P_i, \tag{6}$$

where P_i denotes the probability that input vector $\mathbf{x}$ belongs to region R_i [1]

$$P_i = \int_{R_i} f(\mathbf{x})d\mathbf{x}. \tag{7}$$

As we cited in Introduction, we consider possibility of a near optimum quantizer design applying asymptotic analysis that supposes the piecewise uniform approximation for the joint PDF of two independent Gaussian variables. Therefore, in this section we assume that $f(\mathbf{x})$ is constant f_i over the whole ith region, i.e.

$$P_i = f_i S_i. \tag{8}$$

As a result of this assumption application and after the substitution of (5) in (6), we obtain that the granular distortion per dimension becomes

$$D_g = \frac{1}{12}\sum_{i=1}^{N_q} \frac{f_i S_i^2}{N_i}. \tag{9}$$

This means that N_i should be determined by optimizing the expression (9) in respect to N_i, simultaneously taking into the consideration the constraint given by

$$\sum_{i=1}^{N_q} N_i = N. \tag{10}$$

We perform this optimization by means of Lagrange multiplicator method [2] and we find that (9) has minimal value for

$$N_i = N\frac{f_i^{\frac{1}{2}} S_i}{\sum_{k=1}^{N_q} f_k^{\frac{1}{2}} S_k}. \tag{11}$$

The performance of the proposed quantizer is determined by the granular and the overload distortions. By substituting (11) in (9), similarly as it has been done in [4], we obtain the following expression for the granular distortion

$$D_g^{[4]} = \frac{1}{12}\left(\sum_{i=1}^{N_q} f_i^{\frac{1}{2}} S_i\right)^2. \tag{12}$$

However, observing problem carefully one can note that (12) does not represent objective distortion measure. Namely, the granular distortion determining should be started with (6), where P_i should be calculated exactly by substituting (1) in (7)

$$P_i = \exp\left(-\frac{r_{i-1}^2}{2}\right) - \exp\left(-\frac{r_i^2}{2}\right). \tag{13}$$

Then, substituting (5) and (11) into (6), we obtain that the proposed quantizer has the granular distortion per dimension

$$D_g = \frac{1}{12N}\sum_{k=1}^{N_q} f_k^{\frac{1}{2}} S_k \sum_{i=1}^{N_q} \frac{P_i}{f_i^{\frac{1}{2}}}, \tag{14}$$

where P_i is defined by (13). For the overload distortion calculating, the known expression for the overload distortion of the geometric vector quantizer is used [1]

$$D_o = \frac{1}{2}\int_{r_{N_q}}^{\infty}\left[\left(r - r_{N_q}\right)^2 + \frac{1}{6}\Delta_{N_q}^2\right] f_r\left(r\right) dr, \tag{15}$$

where $f_r(r)$ is the Rayleigh PDF of the vector amplitude ($r\cdot\exp(-r^2/2)$). As a result, we find signal to quantization noise ratio SQNR of our quantizer as

$$\mathrm{SQNR} = 10\log\frac{1}{D_g + D_o}, \tag{16}$$

where D_g and D_o are defined by (14) and (15), respectively.

The obtained results we compare with known distortion minimum ascertained for two-dimensional Gaussian quantization [7]

$$D^{\min} = \frac{2\pi}{3N}, \tag{17}$$

that is with optimal signal to quantization noise ratio

$$\mathrm{SQNR}^{\mathrm{opt}} = 10\log\frac{3N}{2\pi}. \tag{18}$$

III. PROBABILITY DENSITY FUNCTION DISCRETIZATION AND REGION RADIUS DETERMINATION

We perform quantizer designing for the following cases of the PDF approximation

1) Case 1

$$f_i = f\left(\mathbf{x}\right)\Bigg|_{\left(\sum_{k=1}^{2} x_k^2\right) = r_{i-1}^2}. \tag{19}$$

The input vector PDF inside the ith region has the PDF value at lower region boundary.

2) Case 2

$$f_i = f\left(\mathbf{x}\right)\Bigg|_{\left(\sum_{k=1}^{2} x_k^2\right) = r_i^2}. \tag{20}$$

The input vector PDF inside the ith region has the PDF value at higher region boundary.

3) Case 3

$$f_i = f\left(\mathbf{x}\right)\Bigg|_{\left(\sum_{k=1}^{2} x_k^2\right) = \left(\frac{r_{i-1}+r_i}{2}\right)^2}. \tag{21}$$

The input vector PDF inside the ith region has the PDF value at contour whose radius is arithmetic mean of region boundary radii.

4) Case 4

$$f_i = \frac{P_i}{S_i}. \tag{22}$$

The quotient of probability that input vector belongs to the ith region (eq. (13)) and the region surface (eq. (4)) is assumed for the input vector probability density function inside the ith region, as it has been proposed in [2]). Note that expressions (12) and (14) are equal in the last approximation case.

In this paper we examine two examples of the geometric space partition. The first example is presented with the following equation for boundary radius

$$r_i = i\cdot\frac{r_{N_q}}{N_q}, \quad i = 0,\ldots,N_q, \tag{23}$$

where r_{N_q} denotes radius of the support. This example denotes the geometric space partition where the region radii are equidistant. In order to determine the region radii for the other partitioning strategy we introduce a radial scalar compression function $h(r)$. Namely, we note that regarding the symmetry in the cells distribution, the cell size changes only with radial direction, which means that we can define the slope of radial scalar compressor function as

$$\frac{\Delta}{\Delta_i}, \quad i = 1,2,\ldots,N_q, \tag{24}$$

where $\Delta = r_{N_q}/N_q$. If we substitute (5) and (11) in (24), we get that compression function slope is proportional to $f_i^{1/4}$. By further integrating the compression function slope and taking into consideration that compression function should satisfied $h(0)=0$ and $h(r_{N_q}) = r_{N_q}$, we obtain that optimal nonlinear radial compression function for the two-dimensional Gaussian source is

$$h(r) = r_{N_q}\frac{\int_0^r \exp\left(-\frac{t^2}{2}\right)dt}{\int_0^{r_{\max}} \exp\left(-\frac{t^2}{2}\right)dt}. \tag{25}$$

Then, we numerically determine the region radii such that the radial compression function (25) at these radii has the equidistant values

$$r_i \therefore \frac{\int_0^{r_i} \exp\left(-\frac{t^2}{2}\right) dt}{\int_0^{r_{\max}} \exp\left(-\frac{t^2}{2}\right) dt} = \frac{i}{N_q}, \quad i = 0, \ldots, N_q. \tag{26}$$

IV. RESULTS

In this section we present results obtained for the signal to quantization noise ratio of the two-dimensional Gaussian piecewise uniform quantizer designed assuming the geometric space partition and the constant input vector PDF inside regions. We consider two examples of geometric space partition, defined by (23) and (26), as well as four different cases of the input vector PDF approximation, defined by (19) to (22). We will also demonstrate that accuracy of distortion calculation is important for quantizer evaluation. Namely, we calculate signal to quantization noise ratio using (16), (15) and (14), but also as

$$\mathrm{SQNR}^{[4]} = 10 \log \frac{1}{D_g^{[4]} + D_o}, \tag{27}$$

where D_o and $D_g^{[4]}$ are defined with (15) and non-accuracy formula (12), respectively. In Table 1 and 2 we present obtained results.

First, the results we have obtained show that the granular distortion calculation using (12) can lead to wrong conclusions in respect to evaluation of PDF discretization influence on the quantizer design. When the region number is 4, error in signal to quantization noise calculation due to formula (23) application amounts up 3 to 15 dB (see Table 1 and 2). The expectation represents the SQNR calculation for the fourth case of PDF discretization. For this case of quantizer designing, the same results are obtained with (16) and (23). The expectation is also the SQNR calculation for the third case of PDF discretization and the equidistant region boundaries (see the 7th and 8th row in Table I).

Performance estimation using the exact asymptotic formula show that the manner of geometric space partition has significant influence on quantizer designing based on the PDF discretization. For the given quantizer rate and the PDF approximation, comparison of results in Table I and II points out those boundaries of regions should be equidistant. With equidistant region boundaries, the achieved quality is better for 1 to 8 dB depending on the quantizer rate R and the applied PDF approximation.

In Fig. 1, for higher rate, we present the SQNR dependence on the number of regions when the region radii are equidistant.

It is evident that for a small number of regions the manner of discretization has influence on quantizer performance. Namely, only when the number of region is less than 4, it is clearly distincted that the 4th case of PDF approximation is the best. Fig. 1 also shows that for R = 8 bits/sample, the SQNR asymptotically approaches to optimum when the number of regions is 16.

TABLE I. SIGNAL TO QUANTIZATION NOISE RATIO DEPENDENCE ON THE QUANTIZER RATE FOR THE FIRST EXAMPLE OF THE GEOMETRIC SPACE PARTITION AND THE FOUR DIFFERENT CASES OF THE INPUT VECTOR PDF APPROXIMATION. NUMBER OF REGIONS IS $N_Q = 4$.

***R* [bit/sam.]**	5	5.5	6	6.5	7	7.5	8
$r_{\max}$	3.65	3.88	4.09	4.29	4.47	4.63	4.79
$\mathrm{SQNR}^{1[4]}$ [dB]	23.92	26.61	29.35	32.13	34.95	37.80	40.65
SQNR^{1} [dB]	26.68	29.57	32.50	35.44	38.39	41.35	44.31
$\mathrm{SQNR}^{2[4]}$ [dB]	30.11	33.39	36.64	39.87	43.08	46.25	49.43
SQNR^{2} [dB]	26.50	29.34	32.21	35.09	37.98	40.88	43.79
$\mathrm{SQNR}^{3[4]}$ [dB]	26.81	29.75	32.72	35.69	38.67	41.66	44.65
SQNR^{3} [dB]	26.74	29.65	32.59	35.53	38.49	41.45	44.42
$\mathrm{SQNR}^{4[4]}$ [dB]	26.76	29.67	32.61	35.56	38.52	41.49	44.47
SQNR^{4} [dB]	26.76	29.67	32.61	35.56	38.52	41.49	44.47
SQNR^{opt} [dB]	26.89	29.90	32.91	35.92	38.93	41.94	44.95

TABLE II. SIGNAL TO QUANTIZATION NOISE RATIO DEPENDENCE ON THE QUANTIZER RATE FOR THE SECOND EXAMPLE OF THE GEOMETRIC SPACE PARTITION AND THE FOUR DIFFERENT CASES OF THE INPUT VECTOR PDF APPROXIMATION. NUMBER OF REGIONS IS $N_Q = 4$.

***R* [bit/sam.]**	5	5.5	6	6.5	7	7.5	8
$r_{\max}$	3.32	3.5	3.67	3.83	3.99	4.15	4.3
$\mathrm{SQNR}^{1[4]}$ [dB]	20.51	22.67	24.90	27.20	29.53	31.89	34.30
SQNR^{1} [dB]	25.39	28.00	30.64	33.32	36.01	38.72	41.45
$\mathrm{SQNR}^{2[4]}$ [dB]	31.73	35.10	38.40	41.61	44.84	48.09	51.25
SQNR^{2} [dB]	23.99	26.10	28.17	30.20	32.15	34.00	35.84
$\mathrm{SQNR}^{3[4]}$ [dB]	26.43	29.37	32.35	35.38	38.45	41.58	44.72
SQNR^{3} [dB]	25.75	28.41	31.10	33.80	36.51	39.24	41.97
$\mathrm{SQNR}^{4[4]}$ [dB]	25.76	28.44	31.13	33.85	36.59	39.35	42.11
SQNR^{4} [dB]	25.76	28.44	31.13	33.85	36.59	39.35	42.11
SQNR^{opt} [dB]	26.89	29.90	32.91	35.92	38.93	41.94	44.95

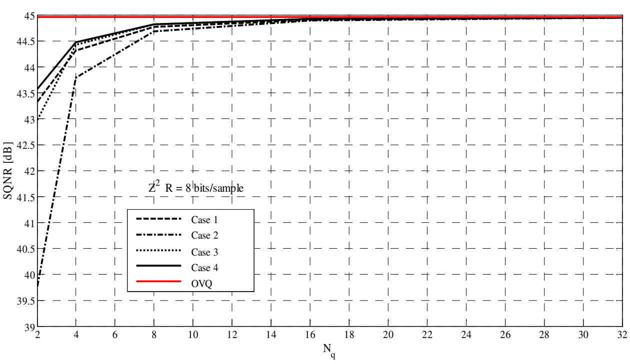

Fig. 1. Signal to quantization noise ratio SQNR as a function of region number N_q for bit number per sample 8 and for the first example of the geometric space partition.

V. CONCLUSIONS

Research conducted in this paper shows that a near optimum two-dimensional Gaussian quantizer can be designed performing PDF discretization and assuming equidistant region boundaries. For a small number of regions (4 at R = 5 bits/sample, i.e. 8 at R = 8 bits/sample), the presented geometric piecewise uniform lattice vector quantization achieves the SQNR that ranges within 0.1 dB to 0.2 dB of the optimum SQNR value.

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