# Improved Fractional Channel Estimation for MIMO Systems in the Low SNR Regime

Simranjit Singh[1], Rajesh Khanna[2], Manjeet Singh Patterh[3]
[1]*UCOE, Punjabi University,*
*Patiala, Punjab, India–147002, phone: +919815966044*
[2]*Thapar University*
[3]*Punjabi University*
*simranjit@live.com*

***Abstract*—In this paper, we propose a novel fractional channel estimation technique for block faded MIMO systems. The existing channel estimation techniques perform the estimation in the time domain which is appropriate for acceptable SNR levels. At low SNR levels, noise which is a major source of errors cannot be separated effectively in time domain. However, fractional domain processing of signals has been shown to be effective in minimizing the errors due to noise. The signal which appears scattered in time domain due to noise appears to be compact in the optimum fractional domain where the effects of noise can be minimized. Based on this principle, a novel channel estimation algorithm is proposed for the low SNR applications. It is seen that the proposed technique clearly outperforms the existing channel estimation technique.**

***Index Terms*—Channel estimation, signal to noise ratio, FRFT, MIMO.**

## I. INTRODUCTION

Multiple-input multiple-output (MIMO) systems have recently emerged as one of the significant technical breakthroughs in modern communication because of its ability to provide high spectral efficiency without the need of additional bandwidth [1]. MIMO system relies upon the knowledge of the channel response at the receiver for data detection and decoding. In practice, however, the CSI is never known to the receiver apriori and some form of channel estimation technique has to be used to estimate the channel response [2], [3]. The channel state information is required at the receiver for equalization, use of multilevel modulation schemes etc.; therefore, it is evident that the operation of the system is dependent on the quality of channel estimate. In this paper, the channel is assumed to obey the block fading law in which the channel remains constant over the full duration of a transmission block and changes completely and independently for the next block. For such a system, commonly used channel estimation techniques are least squares (LS) and minimum mean squared error (MMSE). At a reasonably high signal-to-noise ratio (SNR), these techniques are appropriate for channel estimation, but in the low SNR regime where the errors due to noise are the major factor, these techniques give a poor channel estimate. The signal appears to be scattered in the time domain due to the presence of noise. Due to the scattering, the noise and signal overlap and cannot be separated causing errors in the received signal. Recently, fractional Fourier transform based signal processing techniques have become popular [4]–[8]. Fractional Fourier transform offers flexibility to process the received signal in the intermediate domain between the time and frequency domain while having the same computational complexity as ordinary Fourier transform. In the optimum fractional domain, the received signal appears compact and the distortion due to noise can be minimized. Based on this principle, we propose a novel fractional channel estimation technique for the low SNR regime where the errors are dominated by noise.

## II. SYSTEM MODEL

In this paper we consider a MIMO system with $N_T$ transmitting antennas and $N_R$ receiving antennas. It is assumed that the channel remains constant within one transmission block and changes completely and independently for the next block and then remains constant for the duration of that block. Within one block of L symbols, the MIMO model is

$$\mathbf{Y} = \sqrt{\frac{\rho}{N_T}}\mathbf{HX} + \mathbf{N}, \tag{1}$$

where X denotes the L x $N_T$ transmit signal matrix. **H** is the $N_T$ x $N_R$ channel transfer matrix and represents the scattering medium. The elements $h_{ij}$ represent the fading coefficient from the $i^{th}$ transmit antenna to the $j^{th}$ receive antenna. **Y** is the L x $N_R$ receive signal matrix where the column index of this matrix corresponds to the receive antenna and the row index to the time index [9]. **N** is the complex Gaussian noise matrix with independent identically distributed (i.i.d) with zero mean and unit variance ($\sigma_n^2 = 1$). $\rho$ denotes the signal to noise ratio (SNR) for a single receive antenna.

Manuscript received March 19, 2012; accepted May 15, 2012.

## III. CHANNEL ESTIMATION

The transmit symbol matrix X can be decomposed into

$$\mathrm{X}=\mathrm{X}_T+\mathrm{X}_d \tag{2}$$

and the receive symbol matrix into

$$\mathrm{Y}=\mathrm{Y}_T+\mathrm{Y}_d\,, \tag{3}$$

where $\mathrm{X}_T$, $\mathrm{Y}_T$ and $\mathrm{X}_d$, $\mathrm{Y}_d$ are the training symbols and data symbols respectively.

The block length L can be further decomposed into

$$\mathrm{L}=\mathrm{L}_T+L_d\,, \tag{4}$$

where $\mathrm{L}_T$ is the length of training symbols in a block and $\mathrm{L}_d$ is the length of data symbols in a block.

Let us consider normalized matrix $\overline{\mathbf{X}}_T$ which contains the training symbols only. The optimum training symbol design is based on the orthogonality condition, where the training symbols have to be orthogonal to each other in time and space. The orthogonality condition is given by

$$\overline{\mathbf{X}}_T\overline{\mathbf{X}}_T^H=const.\mathbf{I}_{L_T}\,, \tag{5}$$

where const denotes an arbitrary, real, non-zero factor. The orthogonality condition minimizes the variance of estimation error.

Once the training symbols are received at the receiver, the channel state information can be extracted from them in various methods. Two common techniques used for obtaining channel estimates are LS and MMSE:

$$\hat{\mathbf{H}}_{LS}=\sqrt{\frac{\mathrm{N_T}}{\rho}}(\mathbf{X}_T^H\mathbf{X}_T)^{-1}\mathbf{X}_T^H\mathbf{Y}_T\,, \tag{6}$$

$$\hat{\mathbf{H}}_{MMSE}=\sqrt{\frac{\mathrm{N_T}}{\rho}}(\mathbf{X}_T^H\mathbf{X}_T+\sigma_n^2\mathbf{I}_{N_\mathrm{T}})^{-1}\mathbf{X}_T^H\mathbf{Y}_T\,, \tag{7}$$

where $\hat{\mathbf{H}}_{LS}$ is the least squares estimate and $\hat{\mathbf{H}}_{MMSE}$ are the channel estimates obtained by MMSE method. After estimation the channel matrix can be decomposed into

$$\mathbf{H}=\hat{\mathbf{H}}+\tilde{\mathbf{H}}\,. \tag{8}$$

## IV. FRACTIONAL CHANNEL ESTIMATION

The transmitted and received vectors **X** and **Y** are transformed from time domain to the fractional Fourier domain by using the transformation kernel $\mathbf{K}_a$ or $\mathbf{K}_\alpha$ given by [10]–[13]:

$$K_\alpha(t,u)=\begin{cases}\delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi,\\ \delta(t+u), & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi,\\ \sqrt{\dfrac{1-j\cot(\alpha)}{2\pi}}e^{j\left(\frac{u^2+t^2}{2}\right)},\cot(\alpha)-jut\mathrm{cosec}(\alpha), & \\ & \text{if } \alpha \text{ is not a multiple of } 2\pi,\end{cases} \tag{9}$$

where α denotes the angle by which the signal is rotated in time frequency plane and is related to ‘a’ by α=aπ/2. For angle α varying from 0 to 2π, the value of ‘a’ lies from 0 to 4 and it can be shown that the transformation kernel given by $K_\alpha(t,u)$ is periodic with a period 4. FRFT of order α of x (t) is denoted by $X_\alpha(u)$ and is defined as

$$X_\alpha(u)=\int_{-\infty}^{\infty}x(t)K_\alpha(t,u)dt\,. \tag{10}$$

The transformed received ($\mathbf{Y}_a$) and transmitted ($\mathbf{X}_a$) vectors are given by:

$$\begin{cases}\mathbf{Y}_a=F^a\{\mathbf{Y}\},\\ \mathbf{X}_a=F^a\{\mathbf{X}\},\end{cases} \tag{11}$$

where $F^a$ denotes the a^th order FRFT:

$$\begin{cases}\mathbf{Y}_a=\mathbf{Y}_{T_a}+\mathbf{Y}_{d_a},\\ \mathbf{X}_a=\mathbf{X}_{T_a}+\mathbf{X}_{d_a}.\end{cases} \tag{12}$$

In the training phase we consider $\mathbf{X}_{T_a}$ and $\mathbf{Y}_{T_a}$, and during the information phase we consider $\mathbf{X}_{d_a}$ and $\mathbf{Y}_{d_a}$. The channel estimation in this work is done using the MMSE method given by (7). Channel estimation in the a^th domain is given by

$$\hat{\mathbf{H}}_a=\sqrt{\frac{\mathrm{N_T}}{\rho}}\left(\mathbf{X}_{T_a}^H\mathbf{X}_{T_a}+\sigma_n^2\mathbf{I}_{\mathrm{N_T}}\right)^{-1}\mathbf{X}_{T_a}^H\mathbf{Y}_{T_a}\,, \tag{13}$$

where $\hat{\mathbf{H}}_a$ is the channel gain estimate in the a^th fractional domain.

The parameter ‘a’ for the fractional Fourier domain is selected to minimize the following cost function [14]:

$$\begin{aligned}\tilde{J}(\alpha)=&\sum_i E\left\{[\mathbf{X}_{T_a}]_i[\mathbf{X}_{T_a}^H]_i\right\}-\\ &-2\,\mathrm{Re}\left\{[w_\alpha]_i^H E\left\{[\mathbf{X}_{T_a}]_i[\mathbf{Y}_{T_a}^H]_i\right\}\right\}+\\ &+[w_\alpha]_i[w_\alpha^H]_i E\left\{[\mathbf{Y}_{T_a}]_i[\mathbf{Y}_{T_a}^H]_i\right\},\end{aligned} \tag{14}$$

$$[w_\alpha]_i=\frac{E\left\{[\mathbf{X}_{T_a}]_i[\mathbf{Y}_{T_a}^H]_i\right\}}{E\left\{[\mathbf{X}_{T_a}]_i,[\mathbf{Y}_{T_a}^H]_i\right\}}\,. \tag{15}$$

The angle α is related to the parameter ‘a’ by

$$a=\frac{2\alpha}{\pi}\,. \tag{16}$$

The domain in which the cost function *J* is the optimum domain ‘aopt’ and the channel estimate in the optimum domain can be obtained using (13)

$$\hat{\mathbf{H}}_{aopt} = \sqrt{\frac{\mathrm{N_T}}{\rho}} \left( \mathbf{X}_{T_{aopt}}^{H} \mathbf{X}_{T_{aopt}} + \sigma_n^2 \mathbf{I}_{\mathrm{N_T}} \right)^{-1} \mathbf{X}_{T_{aopt}}^{H} \mathbf{Y}_{T_{aopt}} . \quad (17)$$

After obtaining the channel estimate in the optimum fractional Fourier domain according to the MMSE criteria, it is converted back to time domain by using the inverse FRFT operator ( $F^{-a}$ )

$$\hat{\mathbf{H}}_{opt} = F^{-aopt} (\hat{\mathbf{H}}_{aopt}) . \quad (18)$$

## V. Performance Assessment

In this section, we study the performance of the proposed scheme in the low SNR regime. The total number of transmit and receive antennas is taken to be 4. The block length is taken to be 101 and the total number of blocks is 2000. The training symbols are designed according to the orthogonality condition. The channel is assumed to obey the block fading law in which the channel remains constant over the full duration of a transmission block and changes completely and independently for the next block. The fading is considered to be frequency flat and is assumed to follow Rayleigh distribution. Both the fading channel and noise are comprised of i.i.d complex Gaussian random variables ***CN* (0, 1)**.The value of ‘a’ is varied from -1 to +1 with a step size of 0.1 and 0.01. The results have been taken for four different values of SNR, i.e. -15dB, -10dB, -5dB and 0dB. The step size 0.01 gives better performance but higher computational complexity as compared to step size of 0.1 as step size of 0.01 requires 101 iterations as compared to 0.1 which requires only 21 iterations. The results are presented in the form of mean square error (MSE) vs. ‘a’. The MSE is obtained by the following formula

$$MSE(\boldsymbol{w}_a) = E\left\{ \left\| \mathbf{H} - \hat{\mathbf{H}}_a \right\|^2 \right\} . \quad (19)$$

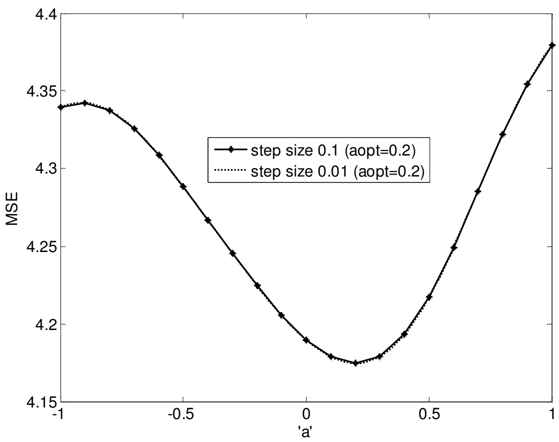


Fig. 1. MSE vs. ‘a’ comparison of existing and proposed receiver for 4x4 MIMO system at a fixed SNR of -15dB.

It is clear from the above results that the channel estimation in the optimum domain gives the minimum mean square error and is therefore better than the conventional time domain channel estimation at ‘a’=0. In this paper, we have considered channel estimation in the low SNR regime.

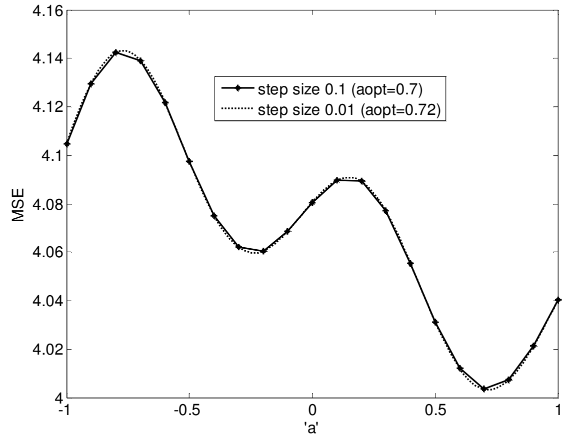


Fig. 2. MSE vs. ‘a’ comparison of existing and proposed receiver for 4x4 MIMO system at a fixed SNR of -10 dB.

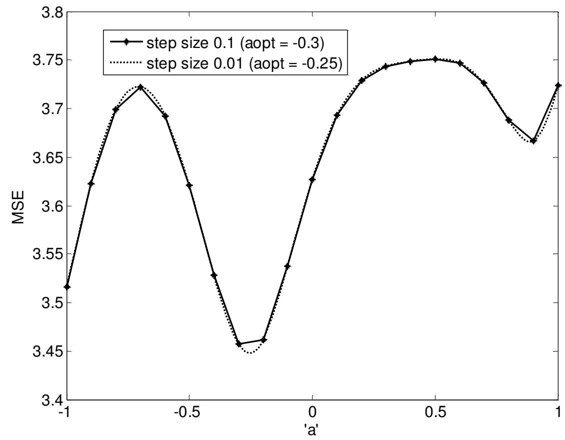


Fig. 3. MSE vs. ‘a’ comparison of existing and proposed receiver for 4x4 MIMO system at a fixed SNR of -5 dB.

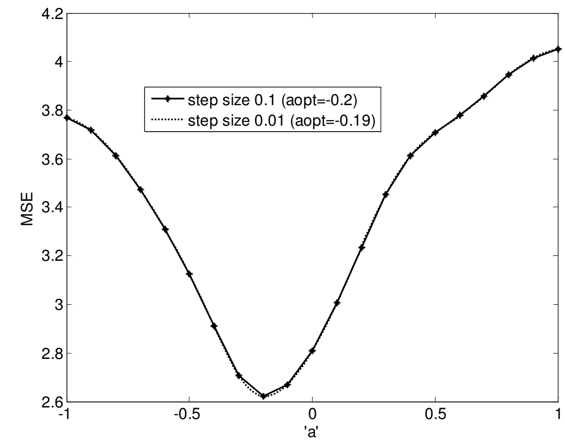


Fig. 4. MSE vs. ‘a’ comparison of existing and proposed receiver for 4x4 MIMO system at a fixed SNR of 0 dB.

Due to high level of distortion due to noise at low SNR, estimation in the fractional domain effectively helps to reduce the effects of noise. In the optimum domain, the signal appears compact which is otherwise scattered by the presence of noise.

## VI. CONCLUSIONS

A novel fractional Fourier domain estimation technique for block faded MIMO systems is proposed in this manuscript. The estimation algorithm is particularly suitable for low SNR applications because the errors at low SNR are dominated by noise and estimation in fractional domain effectively minimizes errors due to noise. From the results it is seen that the proposed channel estimation technique gives better performance than the existing time domain estimation technique. Two different step sizes (0.1 and 0.01) are analyzed and it is found it is more suitable to use a step size of 0.1 due to its lower computational complexity.

## REFERENCES

[1] J. Mietzner, R. Schober, L. Lampe, W. H. Gerstacker, P. A. Hoeher, "Multiple-Antenna Techniques for Wireless Communications – A Comprehensive Literature Survey", *IEEE Communications Surveys & Tutorials,* vol. 11, no. 2, pp. 87–105, 2009. [Online]. Available: http://dx.doi.org/10.1109/SURV.2009.090207

[2] B. Hassibi, B. M. Hochwald, "How Much Training is needed in Multiple-Antenna Wireless Links?", *IEEE Transactions on Information Theory,* vol. 49, no. 4, pp. 951–963, 2003. [Online]. Available: http://dx.doi.org/10.1109/TIT.2003.809594

[3] J. K. Cavers, "An Analysis of Pilot Symbol Assisted Modulation for Rayleigh Fading Channels", *IEEE Transactions on Vehicular Technology,* vol. 40, no. 4, pp. 686–693, 1991. [Online]. Available: http://dx.doi.org/10.1109/25.108378

[4] M. A. Kutay, H. M. Ozaktas, O. Arikan, L. Onural, "Optimal Filtering in Fractional Fourier Domains", *IEEE Transactions on Signal Processing,* vol. 45, no. 5, pp. 1129–1143, 1997. [Online]. Available: http://dx.doi.org/10.1109/78.575688

[5] R. Khanna, R. Saxena, "Improved Fractional Fourier Transform Based Receiver for Spatially Multiplexed MIMO Antenna Systems", *Wireless Personal Communications,* Springer Netherlands, vol. 50, no. 4, pp. 563–574, 2009. [Online]. Available: http://dx.doi.org/10.1007/s11277-008-9637-4

[6] R. Khanna, R. Saxena, "A Novel FRFT Beamformer for Rayleigh Faded Channels", *Wireless Personal Communication,* Springer Netherlands, vol. 52, no. 4, pp. 93–707, 2009.

[7] H. M. Ozaktas, B. Barshan, D. Mendlovic, L. Onural, "Convolution, Filtering and Multiplexing in Fractional Domains and their Relation to Chirp and Wavelet Transforms", *Journal of Optical Society of America-A,* vol. 1, no. 2, pp. 547–559, 1994. [Online]. Available: http://dx.doi.org/10.1364/JOSAA.11.000547

[8] S. Singh, R. Khanna, M. Patterh, "Optimum Combining in Fractional Domain", *International Journal of Electronics,* to be published.

[9] M. Dong, L. Tong, "Optimal Design and Placement of Pilot Symbols for Channel Estimation", *IEEE Transactions on Signal Processing,* vol. 50, pp. 3055–3069, 2002. [Online]. Available: http://dx.doi.org/10.1109/TSP.2002.805504

[10] H. M. Ozaktas, Z. Zalevsky, M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. Wiley, New York, Inc., 2000.

[11] I. S. Yetik, M. A. Kutay, H. Ozaktas, H. M. Ozaktas, "Continuous and discrete fractional Fourier domain decomposition", in *Proc. of the IEEE International Conference on Acoustics, Speech and Signal Processing,* 2000, pp. 93–96.

[12] I. S. Yetik, A. Nehorai, "Beamforming using fractional Fourier transform", *IEEE Transactions on Signal Processing,* vol. 51, no. 6, pp. 1663–1668, 2003. [Online]. Available: http://dx.doi.org/10.1109/TSP.2003.811223

[13] H. M. Ozaktas, O. Arikan, M. A. Kutay, G. Bozdagi, "Digital Computation of the Fractional Fourier Transforms", *IEEE Transactions on Signal Processing,*.vol. 44, no. 9, pp. 2141–2150, 1996. [Online]. Available: http://dx.doi.org/10.1109/78.536672

[14] M. Martone, "A multicarrier system based on the fractional Fourier transform for time-frequency selective channels", *IEEE Transactions on Communication,* vol. 49, no. 6, pp. 1011–1020, 2001. [Online]. Available: http://dx.doi.org/10.1109/26.930631