Entropic Mapping and Green's Function Approximation for Electrostatic Field with Dirichlet Boundary Conditions

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Abstract—**This paper presents entropic updating approach to find the mirror charge, which is induced in the bounded region of the electrostatic field of a single source image. We analyze two sample scenarios with Dirichlet boundary conditions. The first scenario is two spheres with matching centre and the unit charge in between those spheres. The second scenario is a sphere and a plane. Experimental results show that there is no need to have more than *N*=6 steps to find an acceptable Green's function approximation for the electrostatic field of interest. Additionally, the scenario with two spheres shows similar convergence results when compared with eigenfunction expansion.**

Index Terms—**Entropy, green function, electrostatic analysis.**

I. INTRODUCTION

Maximum relative entropy estimation of the vector coordinates using probabilistic approach is consistent with deterministic approach expressions when no uncertainty is present [1]. Entropy maximization leads to the solution which searches for the shortest path (thus, it is an optimization) to the constraints' surfaces. If the entropic updating follows the streamline of the highest gradient, then the question arises whether similar techniques such as conformal mapping exist in entropic updating too. This paper presents three-dimensional (3D) entropic mapping examples where mirror charge maps the source charge in consistent manner within electrostatic field. The selected Dirichlet boundary conditions for two modelling experiments in 3D space are:

1) *Two zero potential spheres;*
2) *A single zero potential sphere and a zero potential plane.*

Finally, we will compare the former (a) case with eigenfunction expansion of Green's function found in the literature [5]. In addition, the consistencies between the sequential probabilistic updating and finding the approximation of Green's function will be discussed.

Recently numerical solutions of the electrostatic field computations lead to the results that are useful for direct current corona field computations [3] and [4], but the modelling had the limitation of two-dimensional (2D) computations. So exploration of analytical methods, that could provide approximated closed form electric field solutions, becomes important.

Moreover, authors expect that such knowledge could potentially expand the means for computations in other related vector fields such as magnetic field too. Recent numerical computation efforts with such simulation software as COMSOL Multiphysics [2] show the possible applications of such approaches.

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II. ENTROPIC MAPPING

In the beginning, we will pick a trivial example of a unit charge and a zero potential sphere as a single Dirichlet boundary condition in 3D space.

The proper relative entropy S will be

$$S = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y,z)\ln\frac{P(x,y,z)}{P_{\text{prior}}(x,y,z)}\mathrm{d}x\mathrm{d}y\mathrm{d}z\,, \tag{1}$$

where x, y and z represent any 3D coordinates on the sphere surface, σ is their deviation and $P(x, y, z)$ is their unknown posterior probability density function (Pdf). Prior distribution P_{prior} and Lagrangian constraints will be:

$$P_{\text{prior}}(x,y,z) = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^3} e^{-\frac{(x-cx)^2}{2\sigma^2}} e^{-\frac{(y-cy)^2}{2\sigma^2}} e^{-\frac{(z-cz)^2}{2\sigma^2}}\,, \tag{2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 1\,, \tag{3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix}(x-xo)^2+(y-yo)^2+\\(z-zo)^2-R^2\end{pmatrix}P(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0\,. \tag{4}$$

where R is the radius of zero potential sphere that exists in 3D electrostatic field, the unit charge is positioned at (cx, cy, cz) and the sphere's centre is at (xo, yo, zo).

By constructing the problem in the manner as in above, we seek the following assumptions: we want to update the charge's position so that the constraint is satisfied, i.e. the unit charge maps on the surface of the sphere.

Maximization of proper relative entropy (1) with constraints (3) and (4) leads to the posterior distribution function, where d is the distance between a unit charge's position at (cx,cy, cz) and the centre of the sphere at (xo, yo, zo) and β is Lagrange multiplier:

$$P(x,y,z)=\sqrt{\left(\frac{1-2\beta\sigma^2}{2\pi\sigma^2}\right)^3}\,e^{k_{x,y,z}}\ , \qquad (5)$$

$$k_{x,y,z}=-\frac{(x-cx)^2+(y-cy)^2+(z-cz)^2}{2\sigma^2}+$$
$$+\beta\left(\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)-\frac{d^2}{1-2\beta\sigma^2}\right), \qquad ,(6)$$

$$\beta=\frac{2R^2-3\sigma^2+\sqrt{9\sigma^4+4R^2d^2}}{4R^2\sigma^2}. \qquad (7)$$

Then estimates fall at the maximum of (5), thus giving:

$$x=xo+\frac{2R^2(cx-xo)}{3\sigma^2+\sqrt{9\sigma^4+4R^2d^2}}, \qquad (8)$$

$$y=yo+\frac{2R^2(cy-yo)}{3\sigma^2+\sqrt{9\sigma^4+4R^2d^2}}, \qquad (9)$$

$$z=zo+\frac{2R^2(cz-zo)}{3\sigma^2+\sqrt{9\sigma^4+4R^2d^2}}. \qquad (10)$$

The value of the source charge uncertainty which produces the estimates that are equal to the coordinates of a unit charge (x=cx, y=cy, z=cz) is

$$\sigma^2_{\text{source}}=\left(R^2-d^2\right)/3. \qquad (11)$$

There is a possibility to shift the sign of the uncertainty (11) value. Inspection of the shifting operation explains the physical meaning of this opposite value of the uncertainty

$$\sigma^2_{\text{image}}=\left(d^2-R^2\right)/3. \qquad (12)$$

This image uncertainty results into the image Pdf which is dual to the source′s Pdf, but with a negative sign.

Applying the value of (12) to (8) – (10) produces:

$$\begin{cases} x=xo+\dfrac{R^2(cx-xo)}{d^2}, \\ y=yo+\dfrac{R^2(cy-yo)}{d^2}, \end{cases} \qquad (13)$$

$$z=zo+\frac{R^2(cz-zo)}{d^2}. \qquad (14)$$

We notice that the expressions (13) and (14) represent the coordinates of the image charge with respect to the source charge, which is present in 3D field with a zero potential sphere. Therefore, there is a direct relationship between these two approaches.

III. Sequential Updating or the Approximation of Green's Function

So far, we found the relationship between the entropic updating and the electrostatic field mapping principles with image charges.

We will try to find the consistency between the sequential updating [6] used in Bayesian analysis and the method of image charges in electrostatics.

Before proceeding with the approximation of Green's function we will remind that a probabilistic impulse response characteristics of a single unit charge in a free 3D space at the time moment t is

$$P_{\text{free}}(x,y,z,cx,cy,cz)=\frac{1}{\left(\sqrt{2\pi t}\right)^3}e^{-\frac{(x-cx)^2+(y-cy)^2+(z-cz)^2}{2t}} \qquad (16)$$

and it corresponds to Brownian motion process with the diffusion coefficient being equal to 1.

Marginalization (integration of the effective period from 0s to infinity implies stationary process) over time t gives Green's function that is

$$G=\frac{1}{2\pi\sqrt{(x-cx)^2+(y-cy)^2+(z-cz)^2}}. \qquad (17)$$

So if we had Green's function approximation or explicit expression consisting of multiple images and their image charges, then Green's function would be, where N is the total number of charges

$$G=\sum_{i=1}^{N}\frac{1}{2\pi\sqrt{(x-cx_i)^2+(y-cy_i)^2+(z-cz_i)^2}}. \qquad (18)$$

We have to mention two important convergence related notes when analyzing (18) and the image charges. First, if a charge is very far away from the region of interest, its resulting Green's function will have values very close to zero and it will have minimal impact to overall result of (18). Second, recall from electrostatics that the value of image charge is equal to $q_{\text{image}}=-Rq_{\text{source}}/d$ when the zero potential sphere is present and the image charge is located in the inside of the sphere. The further the source image is the smaller value the image charge has. When better convergence is required, this dependency occurs in some boundary conditions very useful.

The following is a proposed approach of entropic mapping using the sequential updating or the method of images. A general iterative formula for Green's function when two Dirichlet boundary conditions are present is

$$G=\frac{q_0}{2\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}+$$
$$+\frac{q_{1,1}}{2\pi\sqrt{(x-x'_{1,1})^2+(y-y'_{1,1})^2+(z-z'_{1,1})^2}}+$$
$$+\frac{q_{1,2}}{2\pi\sqrt{(x-x'_{1,2})^2+(y-y'_{1,2})^2+(z-z'_{1,2})^2}}+$$
$$+\sum_{i=2}^{N}\left(\begin{array}{c}\dfrac{q_{i,1}}{2\pi\sqrt{(x-x'_{i,1})^2+(y-y'_{i,1})^2+(z-z'_{i,1})^2}}+\\ \dfrac{q_{i,2}}{2\pi\sqrt{(x-x'_{i,2})^2+(y-y'_{i,2})^2+(z-z'_{i,2})^2}}\end{array}\right), \qquad (19)$$

where $x'_{i,1}$, $y'_{i,1}$, $z'_{i,1}$ are the coordinates of charge and $q_{1,1}$ is the value of the charge induced in the first bounded region, $x'_{i,2}$, $y'_{i,2}$, $z'_{i,2}$ are the coordinates of charge and $q_{1,2}$ is the value of the charge induced in the second bounded region, and i is the index of the step, whereas N is the number of overall steps. The number N should be selected such that the algorithm meats an acceptable convergence by monitoring the criteria which we defined earlier.

IV. Two Spherical Boundary Conditions with Matching Centres

Assume a situation of two spheres with matching centres at $x_{\text{sphere},1}=x_{\text{sphere},2}=10$, $y_{\text{sphere},1}=y_{\text{sphere},2}=10$ and $z_{\text{sphere},1}=z_{\text{sphere},2}=1$. For simplicity sake, we assume a certain relative distance dimension. Also in graphs we assume a projection view onto the plane $\{z-1=0\}$. A unit charge is placed in between surfaces of sphere at $x_0=12$, $y_0=12$, $z_0=1$. The first sphere's radius is $R_1=1$ and the second sphere's radius is $R_1=5$. The total number of steps performed during the computations is $N=6$.

The resulting induced image charges (Table I) show that the values of induced charges decrease quickly with each step.

TABLE I. THE VALUES OF INDUCED CHARGES IN TWO SPHERES.

Index	Charge *q*, C	*x*	y
1	-0.354	10.25	10.25
2	-1.768	16.25	16.25
3	0.2	10.08	10.08
4	5	60	60
5	-0.0707	10.01	10.01
6	-8.84	166.3	166.3
7	0.04	10	10
8	25	1260	1260
9	-0.01414	10	10
10	-44.2	3916	3916
11	0.008	10	10
12	125	31260	31260

Here odd indices represent the charges induced in the first bounded region, while the even indices represent the charges induced in the second bounded region.

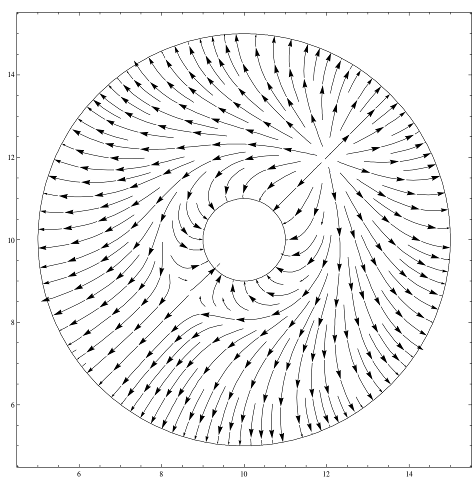

Fig. 1. Projection of vector field streamlines with two spheres.

The layout of the potential field streamlines is present in Fig. 1. There is no need to iterate even further unless we want to reduce the overall computational error even further.

We can see that the potential lines match zero potential values at the boundaries (x=5 and x=15) in Fig. 2.

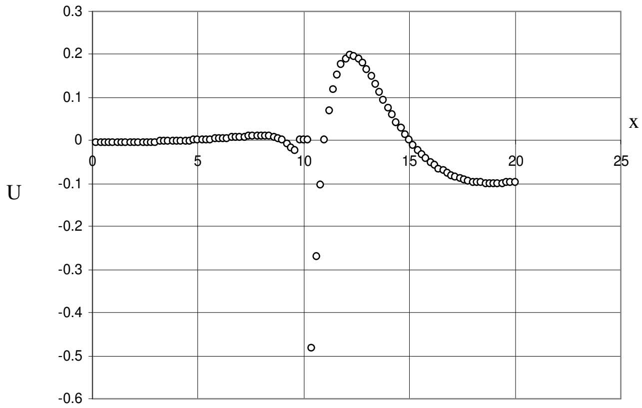

Fig. 2. Potential field curve U (relative dimension) at projection z=1 and y=10.

The experimental results show that it is enough to perform the search for mirror image N=6 times to get an acceptable view of the potential field streamlines.

It appears that the effective potential field on the region of interest (between two spheres) resulting from the remote induced charges becomes less effective with each iteration. Lastly, according to Green's function (17) we can see that the newly induced charges after each step go away much quicker than the previously induced charge's values (Table I).

Comparing these results with literature example on two spheres and eigenfunction expansion [5], where N=10 instances were used; we find that the method of mirror images is still competitive and has similar convergence behaviour.

V. Spherical and Planar Boundary Conditions

We further pick another example where a zero potential sphere (x=10, y=10, z=1, R=5) and a zero potential plane (y=0) are present in the field of a unit charge positioned at $x_0=12$, $y_0=12$, $z_0=1$.

The generated induced image charges are listed in Table II.

TABLE II. THE VALUES OF INDUCED CHARGES IN A SPHERE AND PLANE.

Index	Charge *q*, C	*x*	y
1	-0.41	11.68	11.17
2	-1	20	-17
3	0.1737	10.3	9.19
4	0.41	11.68	-11.17
5	-0.096	10.09	8.83
6	0.1737	10.3	-9.19
7	0.0453	10.02	8.7
8	-0.096	10.09	-8.83
9	-0.0256	10.01	8.67
10	0.0453	10.02	-8.7
11	0.0121	10	8.66
12	-0.0256	10	-8.67

Odd indices represent charges induced in the sphere, and even indices represent charges induced in the plane.

Fig. 3 shows that the layout of 3D electrostatic field is consistent with the expected view.

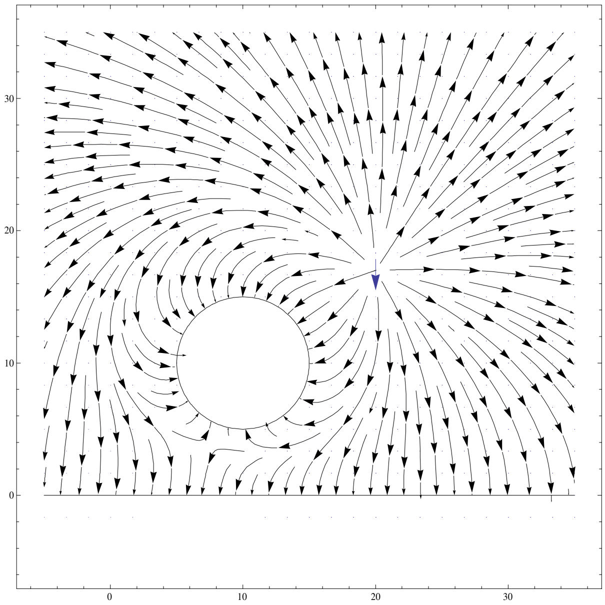

Fig. 3. Projection of vector field streamlines with sphere and plane.

The experimental results show that it is enough to perform the search for mirror image N=6 times to get an acceptable view of the potential field streamlines for the case of one sphere and one plane too.

VI. CONCLUSIONS

This paper presents the relationship between the entropic updating and the method of image charges from electrostatics. A derivation of field equations for a unit charge with a single spherical Dirichlet boundary condition shows the consistency of this approach with sequential updating principles as used in Bayesian updating.

We experimentally research two sample scenarios, which are distinguished by two different convergence criteria. First scenario is when a unit charge is present with two zero potential spheres with matching centres. The second scenario is when a unit charge is present with one zero potential sphere and a plane.

The results show that the method of image charges is competitive with such approaches as eigenfunction expansion [5]. The developed approximation scheme allows finding closed form Green's function approximation suitable for efficient computation of the 3D electrostatic field.

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