http://dx.doi.org/10.5755/j01.eee.19.1.3245 ELEKTRONIKA IR ELEKTROTECHNIKA, ISSN 1392-1215, VOL. 19, NO. 1, 2013

Dynamics Assignment by Output Feedback

K. H. Kiritsis[1]
[1]*Hellenic Airforce Academy Dekelia,*
Kavalas 53, 14231 Nea Ionia, Athens, Greece, phone: +30 210 2779143
costas.kyritsis@gmail.com

***Abstract*—The limits of constant output feedback in altering the dynamics of linear time – invariant systems are studied. In particular an explicit necessary condition is given for a list of polynomials to be invariant polynomials of the closed – loop system. Our method is based on properties of polynomial matrices.**

***Index Terms*—Invariant polynomials, output feedback, polynomial matrices, reachability indices.**

I. INTRODUCTION

In this paper, we shall consider linear discrete- time, completely reachable and observable time-invariant systems. We are interested in determining the limits of constant output feedback in assigning a specified dynamics to the closed-loop system. This problem is called here dynamics assignment by constant output feedback and is probably one of the most prominent open questions in linear system theory. The best available result on dynamics assignment by dynamic output feedback is a sufficient condition given in [1]. The condition consists of inequalities which involve the observability index and reachability indices of the open – loop system and the degrees of the desired invariant polynomials of the closed - loop system. The method of [1] is more general but no necessary and sufficient condition for solvability has obtained as yet.

It is pointed out that the problem of dynamics assignment by state feedback is completely solved in [2], alternative proofs can be found in [3]–[5]. In particular, in [2] a necessary and sufficient condition is given for a list of polynomials to be invariant polynomials of a linear system obtained by state feedback from the given linear system. The condition consists of inequalities which involve the reachability indices of the open – loop system and the degrees of the invariant polynomials of the closed – loop system. In this paper an explicit necessary condition is given for a list of polynomials to be invariant polynomials of a system obtained by constant output feedback from the given linear time invariant system.

II. PROBLEM STATEMENT

Consider a linear discrete - time, left invertible, strictly proper completely reachable and observable linear time-invariant system described by the following equations:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \quad (1)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \quad (2)$$

Manuscript received January 27, 2011; accepted April 4, 2012.

with rank $\mathbf{B}$ = m and the control low

$$\mathbf{u}(k) = -\mathbf{F}\mathbf{y}(k) + \mathbf{v}(k), \quad (3)$$

where $\mathbf{F}$ is an mxp real matrix, $\mathbf{v}(k)$ is the reference input vector of dimensions mx1, $\mathbf{x}(k)$ is the state vector of dimensions nx1, $\mathbf{u}(k)$ is the vector of inputs of dimensions mx1 and $\mathbf{y}(k)$ is the vector of outputs of dimensions px1 and $\mathbf{A}$,$\mathbf{B}$ and $\mathbf{C}$ are real matrices of dimensions nxn, nxm and nxp respectively. By applying the constant output feedback (2) to the system (1) the state equations of closed – loop system are:

$$\mathbf{x}(k+1)=(\mathbf{A}\text{-}\mathbf{BFC})\,\mathbf{x}(k) + \mathbf{B}\mathbf{v}(k), \quad (4)$$
$$\mathbf{y}(k)=\mathbf{C}\mathbf{x}(k). \quad (5)$$

The dynamics of system (4) can be fully described by the invariant polynomial of the polynomial matrix $\mathbf{I}z-$ $-\mathbf{A}+\mathbf{BFC}$. Let $c_1(z)$, $c_2(z)$….$c_m(z)$ be arbitrary monic polynomials over R[z] which satisfy the conditions:

$$c_{i+1}(z) \text{ divides } c_i(z),\ i=1,2,\ldots..m-1, \quad (6)$$
$$\sum_{i=1}^{m} \deg c_i(z) = n\,. \quad (7)$$

The dynamics assignment problem considered in this paper can be stated as follows. Does there exists a constant output feedback (3) such that the system (4) has invariant polynomials $c_1(z)$, $c_2(z)$, …..$c_m(z)$? If so, give conditions for existence.

III. BASIC CONCEPTS AND PRELIMINARY RESULTS

Let us first recall some notions that will be frequently used throughout the paper. Let R be the field of real numbers also let R[z] be the ring of polynomials with coefficients in R. Let $\mathbf{D}(z)$ be a nonsingular matrix over R[z] of dimensions mxm write $\deg_{ci}$ for the degree of column i of $\mathbf{D}(z)$. if

$$\deg_{ci} \mathbf{D}(z) \geq \deg_{cj} \mathbf{D}(z),\ i < j, \quad (8)$$

then the matrix $\mathbf{D}(z)$ is said to be column degree ordered.

Denote $\mathbf{D}_n$ the highest column degree coefficient matrix of $\mathbf{D}(z)$. The matrix $\mathbf{D}(z)$ is said to be column reduced if the real matrix $\mathbf{D}_n$ is nonsingular. A polynomial matrix $\mathbf{U}(z)$ of dimensions kxk is said to be unimodular if and only if has polynomial inverse. The matrix $\mathbf{D}(z)$ is said to be column monic if its highest column degree coefficient matrix is the identity matrix.

Two polynomial matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ having the same numbers of columns are said to be relatively right prime if and only if there are matrices $\mathbf{X}(z)$ and $\mathbf{Y}(z)$ over $R[z]$ such that

$$\mathbf{X}(z)\mathbf{A}(z) + \mathbf{Y}(z)\mathbf{B}(z)=\mathbf{I}, \tag{9}$$

where $\mathbf{I}$ is the identity matrix of dimension rxr, r is the number of columns of the polynomial matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$.

Let $\mathbf{D}(z)$ be a nonsingular matrix over $R[z]$ of dimensions mxm. Then there exist unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ such that

$$\mathbf{D}(z)=\mathbf{U}(z)\ \mathrm{diag}\ [a_1(z), a_2(z), \ldots a_m(z)]\mathbf{V}(z) \tag{10}$$

where the polynomials $a_i(z)$ for i=1,2,…..m are termed the invariant polynomials of $\mathbf{D}(z)$ and have the following property

$$a_i(z) \text{ divides } a_{i+1}(z),\ i=1,2,\ldots m-1. \tag{11}$$

Furthermore we have that

$$a_i(z) = \frac{d_i(z)}{d_{i-1}(z)},\ \ i = 1,2,\ldots\ldots m\ , \tag{12}$$

where $d_o(z)=1$ by definition and $d_i(z)$ is the monic greatest common divisor of all minors of order i in $\mathbf{D}(z)$, i=1,2,…..m. The relationship (10) is known as the Smith – McMillan form of $\mathbf{D}(z)$ over $R[z]$.

Relatively right prime polynomials matrices $\mathbf{D}(z)$ and $\mathbf{N}(z)$ of dimensions mxm and pxm respectively with $\mathbf{D}(z)$ to be column reduced and column degree ordered such that

$$\mathbf{C}(\mathbf{I}z-\mathbf{A})^{-1}\mathbf{B} = \mathbf{N}(z)\mathbf{D}^{-1}(z) \tag{13}$$

are said to form a standard right matrix fraction description of system (1). The column degrees of the matrix $\mathbf{D}(z)$ are the reachability indices of system (1). The system (1) is said to be left invertible if and only if its transfer function matrix (13) has full column rank.

The following lemmas are needed to prove the main theorem of this paper.

A. Lemma 1 [6]

Let $\mathbf{D}(z)$, $\mathbf{N}(z)$ be a standard right matrix fraction description of system (1). Also let v_i for i=1,2,….m be the reachability indices of (1). Then for every mxp real matrix $\mathbf{F}$ we have:

1) The polynomial matrices $\mathbf{N}(z)$ and $[\mathbf{D}(z)+\mathbf{F}\mathbf{N}(z)]$ are relatively right prime over $R[z]$;

2) The matrix $[\mathbf{D}(z)+\mathbf{F}\mathbf{N}(z)]$ is column reduced and column degree ordered and its column degrees are the numbers v_i for i=1,2,…m;

3) The open-loop system (1) and the closed – loop system (4) have the same reachability indices.

Proof. Let $\mathbf{D}(z)$ and $\mathbf{N}(z)$ be a standard matrix fraction description of (1). Then for the transfer function of closed – loop system (4), (5) we have that

$$\mathbf{C}[\mathbf{I}z-\mathbf{A}+\mathbf{BFC}]^{-1}\mathbf{B} = \mathbf{N}(z)\ [\mathbf{D}(z)+\ \mathbf{FN}\ (z)]^{-1} \tag{14}$$

We can write

$$\begin{bmatrix} \mathbf{N}\ (z) \\ \mathbf{D}(z)+\ \mathbf{FN}(z) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ \mathbf{F} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{N}(z) \\ \mathbf{D}\ (z) \end{bmatrix}. \tag{15}$$

Since the matrix

$$\begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ \mathbf{F} & \mathbf{I}_m \end{bmatrix} \tag{16}$$

is unimodular and the matrices $\mathbf{N}(z)$ and $\mathbf{D}(z)$ are relatively right prime over $R[z]$, we have from (15) that the matrices $[\mathbf{D}(z)+\ \mathbf{FN}(z)]$ and $\mathbf{N}(z)$ are relatively right prime over $R[z]$. This is condition (a)of the Lemma.

Since the open – loop system (1) is strictly proper with reachability indices v_i for 1 = 1, 2 …, m we have that

$$\deg_{ci}\mathbf{N}(z) < \deg_{ci}\mathbf{D}(z) = v_i,\ i=1,2,\ldots, m. \tag{17}$$

Since $\mathbf{F}$ is real matrix it follows that from (17)

$$\deg_{ci}\mathbf{FN}(z) < \deg_{ci}\mathbf{D}(z) = v_i,\ i= 1,2,\ldots, m. \tag{18}$$

Since by definition the matrix $\mathbf{D}(z)$ is column reduced and column and degree ordered, it follows from (18) that the matrix $[\mathbf{D}(z)+\ \mathbf{FN}(z)]$ is column reduced and column degree ordered. This is condition (b) of the Lemma.

Since the matrices $[\mathbf{D}(z)+\ \mathbf{FN}(z)]$ and $\mathbf{N}(z)$ are relatively right prime over $R[z]$ and the matrices $\mathbf{D}(z)$ and $[\mathbf{D}(z)+\ \mathbf{FN}(z)]$ are column reduced with the same column degrees, we conclude that the open – loop system (1) and the closed – loop system (4) have the same reachability indices. This is condition (c) of the Lemma and proof is complete.

B. Lemma 2 [3]

Let $\mathbf{D}(z)$, $\mathbf{N}(z)$ be a standard right matrix fraction description of system (1). Then for every mxp real matrix $\mathbf{F}$ the polynomial matrices $[\mathbf{I}z - \mathbf{A}+\mathbf{BFC}]$ and $[\mathbf{D}(z)+\mathbf{FN}(z)]$ have the same nonunit invariant polynomials.

Proof. Let $\mathbf{D}(z)$, and $\mathbf{N}_1(z)$ are relatively right prime polynomial matrices over $R[z]$ of respective dimensions mxm and nxm such that

$$[\mathbf{I}z-\mathbf{A}]^{-1}\ \mathbf{B} = \mathbf{N}_1\ (z)\mathbf{D}^{-1}(z)\ . \tag{19}$$

We have that

$$[\mathbf{I}z-\mathbf{A}]\ \mathbf{N}_1(z) = \mathbf{BD}(z)\ . \tag{20}$$

We add $\mathbf{BFCN}_1(z)$ to both sides of the above identity and rearrange to get

$$[\mathbf{I}z-\mathbf{A}+\mathbf{BFC}]^{-1}\mathbf{B} = \mathbf{N}_1(z)[\mathbf{D}\ (z)+\ \mathbf{FCN}_1\ (z)]^{-1}. \tag{21}$$

Since $[\boldsymbol{I}z-\ \boldsymbol{A}]$ and $\mathbf{B}$ are relatively left prime over $R[z]$ by reachability of (1) and since

$$[\mathbf{I}z-\ \mathbf{A}+\mathbf{BFC}, \mathbf{B}] = [\mathbf{I}z-\mathbf{A}, \mathbf{B}] \begin{bmatrix} \mathbf{I}_n & \mathbf{O} \\ \mathbf{FC} & \mathbf{I}_m \end{bmatrix} \tag{22}$$

it follows that $[\mathbf{I}z-\mathbf{A}+\mathbf{BFC}]$ and $\mathbf{B}$ are relatively left prime over $R[z]$. On the other hand $\mathbf{D}(z)$ and $\mathbf{N}_1(z)$are relatively right prime over $R[z]$ and

$$\begin{bmatrix} \mathbf{N}_1(z) \\ \mathbf{D}(z) + \mathbf{FCN}_1(z) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \boldsymbol{O} \\ \mathbf{FC} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{N}_1(z) \\ \mathbf{D}(z) \end{bmatrix}. \tag{23}$$

Hence $[\mathbf{D}(z) + \mathbf{FCN}_1(z)]$ and $\mathbf{N}_1(z)$ are relatively right prime over R[z]. It follows that the matrices $[\mathbf{I}z - \mathbf{A} + \mathbf{BFC}]$ and $[\mathbf{D}(z) + \mathbf{FCN}_1(z)]$ or equivalently the matrices $[\mathbf{I}z - \mathbf{A} + \mathbf{BFC}]$ and $[\mathbf{D}(z) + \mathbf{FN}(z)]$ must share the same nonunit invariant polymonials and proof is complete.

IV. Main Results

The main result of the paper is given below and gives an explicit necessary condition for the existence of solution of dynamics assignment problem by constant output feedback.

A. *Theorem 1*

Let $\mathbf{D}(z)$, $\mathbf{N}(z)$ be a standard right matrix fraction description of left invertible system (1) and $\mathbf{n}(z)$ the last column of the polynomial matrix $\mathbf{N}(z)$. Also let $\nu_1 \geq \nu_2 \geq \ldots\ldots \geq \nu_m$ be the ordered list of reachability indices of (1). Then the dynamics assignment problem by constant output feedback has a solution only if:

- the rows of the polynomial vector $\mathbf{n}(z)$ span the linear space over R of all polynomials $\lambda(z)$ over R[z] with $\deg\lambda(z) \leq \nu_m - 1$.

Proof. To establish necessity, suppose that there exists a constant output feedback (3) such that the system (4) has invariant polynomial $c_1(z)$, $c_2(z), \ldots\ldots, c_m(z)$. Then by Lemma 2 these invariant polynomials coincide with the invariant polynomials of the matrix $\mathbf{C}(z)$ given by

$$\mathbf{D}(z) + \mathbf{FN}(z) = \mathbf{C}(z). \tag{24}$$

Without any loss of generality we assume that the matrix $\mathbf{D}(z)$ is column monic (since this can always be achieved by a constant nonsingular input transformation). Then from Lemma 1 it follows that the matrix $\mathbf{C}(z)$ is column monic and column degree ordered and its column degrees are the ordered list of reachability indices of system (1)

$$\deg_{ci}\mathbf{C}(z) = \nu_i, i = 1,2,\ldots, m. \tag{25}$$

Let $\mathbf{a}(z)$ be the last column of the matrix $\mathbf{C}(z)$, also let $a_{im}(z)$ for i=1,2,….,m the elements of $\mathbf{a}(z)$. Since the matrix $\mathbf{C}(z)$ is column monic we have that

$$\deg a_{im}(z) < \nu_m, i = 1,2,\ldots m-1 \text{ and } \deg a_{mm}(z) = \nu_m. \tag{26}$$

Since $c_1(z), c_2(z), \ldots, c_m(z)$ are arbitrary monic polynomials subject, however, to conditions (6) and (7), we assume without any loss of generality that

$$\deg c_i(z) = \nu_i, i = 1,2,\ldots, m. \tag{27}$$

From the relationships (11) and (12) it follows that the polynomial $c_m(z)$ is the greatest common divisor of all elements of matrix $\mathbf{C}(z)$. This implies that

$$c_m(z) \text{ divides } a_{im}(z) \quad i = 1,2,\ldots m. \tag{28}$$

Then from (26) and (28) it follows that

$$a_{im}(z) = 0, \quad i=1,2,\ldots,m-1 \text{ and } a_{mm}(z) = c_m(z). \tag{29}$$

From (29) we have that the polynomial vector $\mathbf{a}(z)$ has the following structure

$$\mathbf{a}^{T}(z) = [0,\ldots,0, c_m(z)]. \tag{30}$$

Let $\mathbf{d}(z)$ be the last column of the matrix $\mathbf{D}(z)$. Then from (24) we have that

$$\mathbf{Fn}(z) = \mathbf{a}(z) - \mathbf{d}(z). \tag{31}$$

Since the matrices $\mathbf{D}(z)$ and $\mathbf{C}(z)$ are both column monic we have that

$$\deg[\mathbf{a}(z) - \mathbf{d}(z)] \leq \nu_m - 1. \tag{32}$$

Since $c_m(z)$ is arbitrary monic polynomial of degree ν_m, the linear space V over R spanned by the rows of polynomial vector $[\mathbf{a}(z)-\mathbf{d}(z)]$, consists of all polynomials $\lambda(z)$ over R[z] with

$$\deg\lambda(z) \leq \nu_m - 1. \tag{33}$$

Since equation (31) has by assumption solution for $\mathbf{F}$ over R, the rows of $\mathbf{n}(z)$ span V. This is condition of the Theorem and proof is complete.

V. Example

Consider system (1) specified by:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{34}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \tag{35}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{36}$$

The task is to check if the problem of dynamics assignment has a solution by constant output feedback.
We form the matrix

$$[\mathbf{I}z - \mathbf{A}, -\mathbf{B}] \tag{37}$$

and reduce it to a lower a triangular form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -z & 1 & 0 & 0 & 0 \\ 0 & 0 & -z & 1 & 0 & 0 \end{bmatrix}, \tag{38}$$

by elementary column operations

The unimodular matrix which represents these operations is

$$\mathbf{U}(z) = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & z & z \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & (z-1) & z^2 & z \\ -1 & 0 & 1 & (1-z) & 0 & (z^2-z-1) \end{bmatrix} \tag{39}$$

We define the matrices:

$$\mathbf{N}_1(z) = \begin{bmatrix} 0 & z \\ 1 & 1 \\ z & z \\ 0 & 1 \end{bmatrix}, \qquad (40)$$

$$\mathbf{D}(z) = \begin{bmatrix} z^2 & z \\ 0 & z^2 - z - 1 \end{bmatrix}, \qquad (41)$$

$$\mathbf{N}(z) = \mathbf{C} \cdot \mathbf{N}_1(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}. \qquad (42)$$

We have that

$$\mathbf{C}(\mathbf{I}z - \mathbf{A})^{-1}\mathbf{B} = \mathbf{N}(z)\mathbf{D}^{-1}(z). \qquad (43)$$

This implies that the matrices $\mathbf{N}(z)$ and $\mathbf{D}(z)$ form a standard right matrix fraction description of system (1). The reachability indices of system (1) are seen to be $\nu_1 = \nu_2 = 2$. The last column $\mathbf{n}(z)$ of the polynomial matrix $\mathbf{N}(z)$ is

$$\mathbf{n}(z) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \qquad (44)$$

It is obvious that the rows of the polynomial vector $\mathbf{n}(z)$ does not span the linear space over R of all polynomial $\lambda(z)$ with $\deg\lambda(z) \leq 1$. This implies that for the given system the problem of dynamics assignment by constant output feedback has no solution

VI. CONCLUSIONS

In this paper the problem of dynamics assignment by constant output feedback is studied. In particular an explicit necessary condition is given for the problem to have a solution. It is pointed out that this problem is very hard and is probably one of the most prominent open questions in linear system theory. The solution of many important unsolved control problems such as pole placement of linear system by dynamic or constant output feedback, dynamics assignment by dynamic output feedback, stabilızation of linear systems by constant output feedback and stabilization of linear systems using minimal order dynamic compensators, strongly depends on the solution of dynamics assignment problem. Some of the above problems will be automatically solved, if the problem of dynamics assignment by constant output feedback would have been solved. The difficulty of the longstanding open problem of dynamics assignment by output feedback comes from its essentially nonlinear nature of the problem structure. It is an intrinsically nonlinear problem in linear system theory. Matrix algebra loses its power in face of nonlinear nature of the problem. In view of far reaching progress of linear system theory it is somewhat surprising to notice that such a fundamental problem is still open.

REFERENCES

[1] G. K. Hayton, H. H. Rosenbrock, “The general Problem of pole Assignment”, *International Journal of Control,* vol. 6, no. 27, pp. 837–852, 1978.

[2] H. H. Rosenbrock, *State–Space and Multivariable Theory.* London: Nelson, 1970, p. 165.

[3] V. Kucera, “Assigning the invariant factors by feedback”, *Kybernetika,* – vol. 2, no. 17, pp. 118–127, 1981.

[4] B. W. Dickinson, “On the fundamental theorem of linear state variable feedback”, *IEEE Transactions on Automatic Control,* vol. 5, no. 19, pp. 577-579, 1974. [Online]. Available: http://dx.doi.org/10.1109/TAC.1974.1100639

[5] D. S. Flamm, “A New Proof of Rosenbrock’s Theorem on Pole Assignment”, *IEEE Transactions on Automatic Control,* no. 6, vol. 25, pp. 1128–1133, 1980. [Online]. Available: http://dx.doi.org/10.1109/TAC.1980.1102509

[6] A. C. Pugh, P. A. Ratcliffe, “Infinite Frequency Interpretations of Minimal Bases”, *International Journal of Control,* no 4, vol. 32, pp. 581–588, 1980. [Online]. Available: http://dx.doi.org/10.1080/00207178008922874