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Christoffel-Darboux Formula Most Directly Applied in Generating Economical Linear Phase Low-Pass Digital FIR Filter Functions

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***Abstract*—The approximation problem of a filter function of even and odd order is solved mathematically in this paper most directly applying the proposed Christoffel-Darboux formula for two continual orthogonal polynomials on the equal finite segment. As a result, a linear phase low-pass digital finite impulse response (FIR) filter function is generated in compact explicit form. In addition, a new difference equation and structure of digital FIR filter are proposed. Two examples of the extremely economic FIR filters (with four adders and without multipliers) designed by the proposed technique are presented. The proposed solutions are efficient in regard to energy consumption and have a high selectivity.**

***Index Terms*—Difference equations, digital filters, filtering theory, finite impulse response filter.**

I. INTRODUCTION

Theory of filtering has wide applications in various frequency ranges and technologies for analog and digital signals [1]. During previous decades of rapid developments in this theory, various techniques have been used for solving complex and always actual problem of generating filter functions. The classical Christoffel-Darboux formula [2], [3] for continual orthogonal polynomials is shown to be an important identity with extremal properties for that purpose. This formula for classical Jacobi orthogonal polynomials and for all their particular solutions (Gegenbauer, Legendre and Chebyshev polynomials of the first and second kind) has been applied for generating new class filter functions [4], [5]. The global Christoffel-Darboux formula for four orthonormal polynomials on two equal finite segments for generating the linear phase two-dimensional FIR digital filter functions has been proposed in a compact explicit form [1]. In addition, successful applications of powerful orthogonal polynomials for a mean-square approximation of the filter magnitude characteristic in the pass-band are well-known [6], [7], while application of orthogonal Chebyshev polynomial in generating all-pole filter function with decreasing envelope of the summed sensitivity function has been described in the literature [8].

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In this paper, the Christoffel-Darboux formula for two orthogonal polynomials on the equal finite segment is proposed in a compact explicit form. This formula can be most directly applied in generating linear phase selective low-pass digital FIR filter functions as demonstrated here. Besides, a new difference equation generated by the proposed Christoffel-Darboux formula for continual signals and corresponding FIR filter function as well as filter structure are proposed. Relevant examples of the linear phase selective low-pass digital FIR filters designed by the proposed technique are presented.

II. MATHEMATICAL BACKGROUND FOR CONTINUAL VARIABLES

Let note a set of continual non-periodical orthogonal polynomial as $P_r(x)$ where x is a continual variable and r is the order of these polynomials on a finite interval $-a \le x \le b$ with respect to the non-negative continual weight function, $w_1(x)$. Orthogonality of this set is defined by

$$\int_{-a}^{b} w_1(x) P_r(x) P_k(x)\,dx = 0 \quad r \neq k;\ r, k = 0,1,2,3,\ldots \tag{1}$$

For the introduced polynomials $P_r(x)$, r-th order norm, $h_1(r)$, is given by

$$h_1(r) = \int_{-a}^{b} w_1(x) \left(P_r(x)\right)^2 dx \quad r = 1,2,3,\ldots \tag{2}$$

Besides, let $Q_r(y)$ be another set of continual non-periodical polynomials, where y is a continual variable, r is the order of the mentioned polynomials on a finite interval $-a \le y \le b$ with respect to the non-negative continual function, $w_2(y)$, and orthogonality defined by

$$\int_{-a}^{b} w_2(y) Q_r(y) Q_k(y)\,dy = 0 \quad r \neq k;\ r, k = 0,1,2,3,\ldots \tag{3}$$

For these polynomials, $Q_r(y)$, r-th order norm, $h_2(r)$ is

$$h_2(r) = \int_{-a}^{b} w_2(y)\left(Q_r(y)\right)^2 dy \qquad r = 1, 2, 3, \ldots \tag{4}$$

The finite (summed from zero to n-th component) Christoffel-Darboux formula for two orthogonal polynomials of the same order, r, with x and y as variables, $P_r(x)$ and $Q_r(y)$, $r = 0, 1, \ldots, n$ (n is the maximum order of continual orthogonal polynomials), on the equal finite segment $[-a, b]$, and corresponding norms $h_1(r)$ and $h_2(r)$ is proposed here in the following explicit compact form of orthonormal components [5]

$$\frac{P_0(x)Q_0(y)}{\sqrt{h_1(0)h_2(0)}} + \frac{P_1(x)Q_1(y)}{\sqrt{h_1(1)h_2(1)}} + \ldots + \\ + \frac{P_k(x)Q_k(y)}{\sqrt{h_1(k)h_2(k)}} + \ldots + \frac{P_n(x)Q_n(y)}{\sqrt{h_1(n)h_2(n)}} \tag{5}$$

or

$$\sum_{r=0}^{n} \frac{P_r(x)Q_r(y)}{\sqrt{h_1(r)h_2(r)}}. \tag{6}$$

The previous formula can be mapped into the new domains, analogue, s, and digital, z, by applying a standard technique [9]. For example, the following relations can be always set in the z domain:

$$T_k \to \cos(k\omega) \to (e^{jk\omega} + e^{-jk\omega})/2 \to (z^k + z^{-k})/2, \tag{7}$$

$$U_k \to \sin(k\omega) \to (e^{jk\omega} - e^{-jk\omega})/(2j) \to \\ \to (z^k - z^{-k})/(2j), \tag{8}$$

where T_k and U_k are the orthogonal Chebyshev polynomials of the first kind and second kind, respectively. Another way of mapping is given by the following example:

$$x^4 = \frac{1}{8}\left[3T_0(x) + 4T_2(x) + T_4(x)\right] \to \\ \to \frac{1}{8}\left[3 + 4\cos(2\omega) + \cos(4\omega)\right] \to \\ \to \frac{1}{16}\left[6 + 4(z^{+2} + z^{-2}) + (z^{+4} + z^{-4})\right], \tag{9}$$

$$x^5 = \frac{1}{16}\left[10T_1(x) + 5T_3(x) + T_5(x)\right] \to \\ \to \frac{1}{16}\left[10\cos(\omega) + 5\cos(3\omega) + \cos(5\omega)\right] \to \\ \to \frac{1}{32}\left[10(z^{+1} + z^{-1}) + 5(z^{+3} + z^{-3}) + (z^{+5} + z^{-5})\right]. \tag{10}$$

Applying the initial Christoffel-Darboux formula for continual variables, (5), and described mappings, (7) to (10), equivalent equations in the z domain can be obtained in a similar way as in two dimensions [1].

III. PROPOSED SELECTIVE LOW-PASS FIR FILTER FUNCTION

A difference equation (derived from (5)) of the following form is proposed here

$$y[n] = y[n-2] + x[n] - x[n-2N] + \\ + x[n-2N-2] - x[n-4N-2], \tag{11}$$

where N is an integer parameter. This equation can be represented in the z domain by a causal low-pass FIR filter function

$$H(N, z) = H(z) = \frac{1 + z^{-2(N+1)}}{1 - z^{-2}}\left(1 - z^{-2N}\right). \tag{12}$$

The normalization constant, K_1, is excluded from the previous equation, while in practice the function $H(N, z)$ is multiplied by K_1. The general structure of the selective low-pass FIR filter whose filter function is given by (12) in recursive realization is presented in Fig. 1. This filter is economical as it has only four adders (without multipliers), and it is based on a set of delays.

The frequency response of the proposed filter is given by

$$H(e^{j\omega}) = \frac{1}{2}e^{-j2N\omega}\frac{\cos\left((N+1)\omega\right)}{\sin(\omega)}\sin(N\omega). \tag{13}$$

As any other complex function, the previous frequency response can be expressed in terms of its magnitude and phase characteristics, respectively:

$$\left|H(e^{j\omega})\right| = \frac{1}{2}\frac{\cos\left((N+1)\omega\right)}{\sin(\omega)}\sin(N\omega), \tag{14}$$

$$\arg\left(H(e^{j\omega})\right) = -2N\omega + 2k\pi, \quad k = 0, 1, 2, \ldots \tag{15}$$

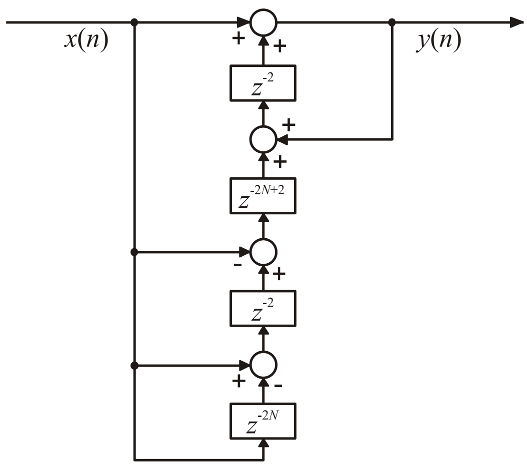


Fig. 1. Structure of the selective low-pass FIR filter defined by (12) in recursive realization.

The last equation shows that the proposed filter has a linear phase, which is manifested by the constant group delay, $\tau(\omega)$, expressed as

$$\tau(\omega) = 2N. \tag{16}$$

According to (12) and (16), it can be concluded that the filter impulse response length is $4N$.

IV. GENERAL FORM OF FILTER FUNCTION

A linear phase low-pass FIR filter, $H(z)$, of order N is defined by the factorized form [9]

$$H(z) = \prod_{r=1}^{N}\left(1 - \frac{z}{z_r}\right), \tag{17}$$

where z_r, $r = 1, 2, \dots, N$ are the filter function zeros circularly symmetric in reference to the unity circle.

Similarly to the approach used in the literature [10], a filter function of K cascaded blocks $H(z)$, given by (12), of the form

$$\left(H(N, z)\right)^K = \left(H(z)\right)^K, \tag{18}$$

representing multiple zero function is considered here. In addition, we propose that the function $H(z)$ is realized as a product of three functions of successive orders $M-1$, M and $M+1$

$$H(M-1, z)H(M, z)H(M+1, z). \tag{19}$$

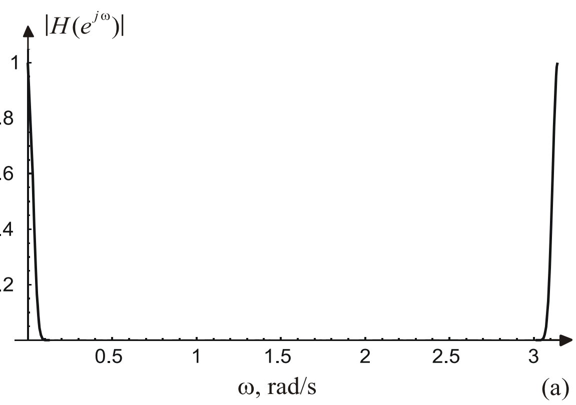
So, general form of the proposed filter function is given by the following expression

$$\left[H(M-1, z)H(M, z)H(M+1, z)\right]^K. \tag{20}$$

V. FILTER EXAMPLES

A. Example 1

An example of a filter is obtained for particular values of free integer parameters K and M, $K = 3$ and $M = 9$ from (20), applying (5). In Fig. 2, the magnitude characteristic of the proposed linear phase selective low-pass FIR filter is shown.



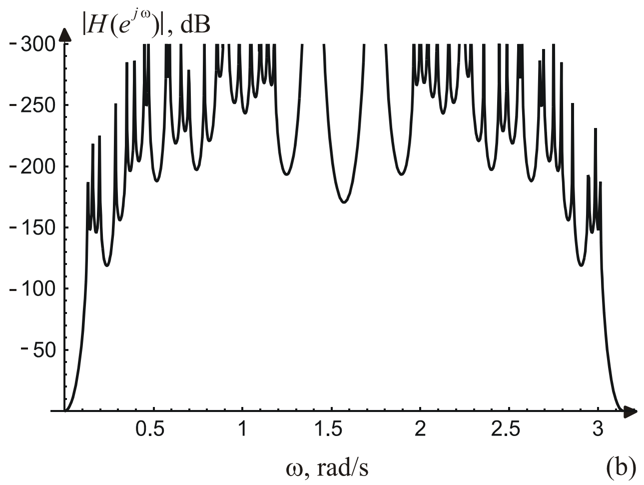
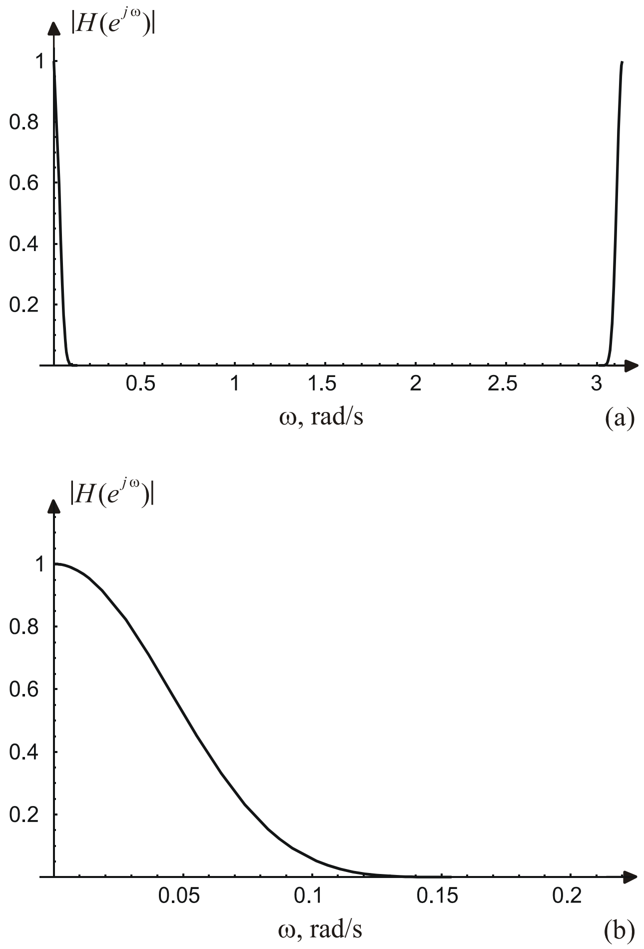


Fig. 2. The magnitude characteristic of the proposed linear phase selective low-pass FIR filter obtained by (5) for $K = 3$ and $M = 9$ from (20): (a) in absolute units and (b) in dBs.

The filter realization is given by the structure presented in Fig. 1.

B. Example 2

Another example of proposed linear phase selective low-pass FIR filter is obtained for particular values of integer parameters K and M, $K = 2$ and $M = 7$ from (20), applying (5) whose magnitude characteristic is shown in Fig. 3. Realization of this filter is also given in Fig. 1.



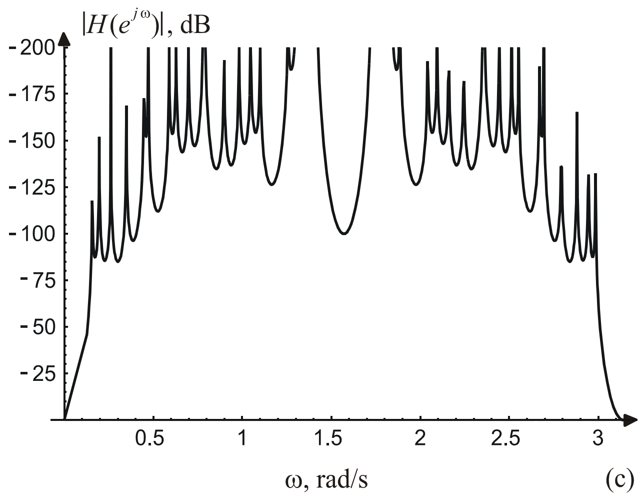


Fig. 3. The magnitude characteristic of the proposed linear phase selective low-pass FIR filter obtained by (5) for $K = 2$ and $M = 7$ from (20): (a) in absolute units, (b) zoomed characteristic in absolute units and (c) characteristic in dBs.

VI. Conclusion

An original approach to a linear phase selective low-pass digital FIR filter design is presented in this paper. It is based on applying the proposed Christoffel-Darboux formula for two continual orthogonal polynomials on the equal finite segment. This formula is appropriate for generating a FIR filter function in compact explicit form. A new difference equation derived by the proposed formula and corresponding causal linear phase selective low-pass FIR filter function as well as filter structure are proposed here.

The filters realized based on this function for any particular value of the parameter N are extremely economic regarding structure and energy consumption since they have only four adders and no mulitpliers, which is a direct consequence of the proposed difference equation form.

Two examples of the FIR filter designed by the proposed approximation technique are illustrated. These efficient low-pass FIR filters have a high attenuation in the stop-band and high selectivity.

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