ISSN 1392 – 1215 ELECTRONICS AND ELECTRICAL ENGINEERING 2012. No. 3(119)
ELEKTRONIKA IR ELEKTROTECHNIKA

T 180 TELECOMMUNICATIONS ENGINEERING
TELEKOMUNIKACIJŲ INŽINERIJA

Efficient Multirate EMQF Digital Filters

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crossref http://dx.doi.org/10.5755/j01.eee.119.3.1360

Introduction

The operating rate of digital programmable chips can be significantly larger than the required processing rate for digital signal processing applications. The efficiency of digital filter implemented on the programmable hardware, such as field programmable gate array (FPGA) chips, can be very low, because in the most of the available operating time the implemented digital filter is not used. The efficiency of the implementation of digital filter on a programmable or dedicated hardware can be considerable improved when using the same hardware more than ones during the sampling period.

A method for improving the efficiency of digital filters using filter sharpening technique has been presented in [1]. Two types of low-order elliptic filters are used several times in order to implement a higher order filter with the same passband and stopband edge frequencies, in such a way that overall filter has a very small passband ripple and may have very large stopband attenuation. The same technique can be used for polynomial approximations [2].

The technique was formerly suggested for continuous-time filter design [3] so that very selective filters can be implemented with low Q factors. Filters with lower Q factors are more robust for implementation and are less sensitive to element changes due to temperature variations and finite element tolerances.

The digital counterpart of elliptic minimum Q factor (EMQF) continuous-time filter is very attractive for hardware implementation because it has smaller number of filter coefficients, and the frequency response is less sensitive to coefficient quantization [4].

In his paper, a new version of efficient digital filter design method is presented. The method is based on the most efficient low-order EMQF filter [4] as a basic sub-filter that can be used several times. The sharpening method uses all implementation benefits suggested in [5–8].

The design of elliptic digital filters is performed using algorithms from [9]. The derivation and proofs are done using computer algebra systems (CAS) in a manner similar to that described in [10].

Design of odd-order elliptic and EMQF filters

The elliptic digital filters are the most efficient filters because there is no other filter of the lower order that can fulfill the filter specification [9]. EMQF filters are the most efficient filters of the elliptic filters [4].

The drawback of the all-pass implementation is that the word-length should be larger for larger stop-band attenuation. Therefore, the low-order EMQF filter with not too large stopband attenuation is the optimal filter structure as the basic building block of filters implemented on FPGA or dedicated hardware with multiple usages during the sampling period. The filter order of EMQF filter should be odd so that the minimal number of coefficients is required for implementation when using the all-pass branches.

Since the requirement for the stop-band attenuation of the basic filter section can be between 10 to 20 dB, the fifth order elliptic filter can be sufficient even for filters with a sharp transition.

The design algorithm is described in [9]. The input parameters are derived from the filter specification: filter order, or the order of the transfer function of digital filter is n, the pass-band edge frequency is f_p, the stop-band edge frequency f_s, and the selectivity factor is ξ

$$\xi(f_p,f_s)=\frac{\tan(\frac{\pi f_s}{2})}{\tan(\frac{\pi f_p}{2})}. \quad (1)$$

The bilinear transformation (BT) transforms the frequency s from the continuous-time domain into the corresponding discrete-time domain for known pass-band edge frequency f_p

$$BT(s, f_p) = \frac{1 + s\tan(\pi f_p)}{1 + s\tan(\pi f_p)}\,. \tag{2}$$

The basic variables for computing the transfer function are the transfer function zeros in the s domain:

$$\begin{cases} X(n,\xi,i) = -\mathrm{cd}\left(\dfrac{2i-1}{n} K_J\left(\dfrac{1}{\xi}\right), \dfrac{1}{\xi}\right), \\ X\left(n,\xi,\dfrac{n+1}{2}\right) = 0, \quad n \text{ odd}, \end{cases} \tag{3}$$

where K_J is the complete elliptic integral of the first kind, and cd is Jacobi elliptic function. The number of zeros is equal to the order of the transfer function, i=1,…,n.

The value of the approximation function at the stop-band edge frequency is called discrimination factor

$$L(n,\xi) = \frac{1}{\xi^{n-2}} \frac{\prod_{i=1}^{\frac{n-1}{2}} \left(\xi^2 - (X(n,\xi,i))^2\right)^2}{\prod_{i=1}^{\frac{n-1}{2}} \left(1 - (X(n,\xi,i))^2\right)^2}, \quad n \text{ odd}. \tag{4}$$

The transfer function poles depend on the ripple factor ε, and an auxiliary function ζ:

$$\begin{cases} \zeta(n,\xi,\varepsilon) = \mathrm{sn}\left(\dfrac{K_J\left(\sqrt{1-\dfrac{1}{\xi^2}}\right)}{K_J\left(\sqrt{1-\dfrac{1}{(L(n,\xi))^2}}\right)} v, \sqrt{1-\dfrac{1}{\xi^2}} \right), \\ v = \mathrm{sn}^{-1}\left(\dfrac{1}{\sqrt{1+\varepsilon^2}}, \sqrt{1-\dfrac{1}{(L(n,\xi))^2}} \right), \end{cases} \tag{5}$$

where sn is the Jacobi elliptic function, and sn^{-1} is the inverse Jacobi elliptic function. Variable v is used for presenting the expression in two rows.

The transfer function poles in the s domain are:

$$\begin{cases} S(n,\xi,\varepsilon,i) = \dfrac{N_{\mathrm{re}} + j\,N_{\mathrm{im}}}{1 - (\zeta(n,\xi,\varepsilon))^2 \left(1 - \dfrac{(X(n,\xi,i))^2}{\xi^2}\right)}, \\ N_{\mathrm{re}} = -\zeta(n,\xi,\varepsilon)\sqrt{1-(\zeta(n,\xi,\varepsilon))^2} \times \\ \qquad \times \sqrt{1-(X(n,\xi,i))^2}\left(1 - \dfrac{(X(n,\xi,i))^2}{\xi^2}\right), \\ N_{\mathrm{im}} = X(n,\xi,i)\sqrt{1-(\zeta(n,\xi,\varepsilon))^2\left(1-\dfrac{1}{\xi^2}\right)}. \end{cases} \tag{6}$$

The odd-order transfer function, $H=H(n,\xi,\varepsilon,f_p,z)$, is generated using the known transfer function zeros, the transfer function poles, and the bilinear transformation:

$$\begin{cases} z_{zi} = BT\left(j\dfrac{\xi}{X(n,\xi,i)}, f_p\right), \\ p_{zi} = BT(S(n,\xi,\varepsilon,i), f_p), \\ g = \dfrac{2\prod_{i=1}^{\frac{n-1}{2}} (1-\mathrm{Re}(z_{zi}))}{(1-z_{p\frac{n+1}{2}})\prod_{i=1}^{\frac{n-1}{2}} (1-2\,\mathrm{Re}(z_{pi}) + |z_{pi}|^2)}, \\ H = \dfrac{\dfrac{z+1}{g}\prod_{i=1}^{\frac{n-1}{2}} (z^2 - 2z\,\mathrm{Re}(z_{zi}) + 1)}{(z-z_{p\frac{n+1}{2}})\prod_{i=1}^{\frac{n-1}{2}} (z^2 - 2z\,\mathrm{Re}(z_{pi}) + |z_{pi}|^2)}. \end{cases} \tag{7}$$

A special case of elliptic filters is the EMQF filter. The main property of the EMQF filter is that the ripple factor is computed using the known discrimination factor

$$\varepsilon_{EMQF} = \frac{1}{\sqrt{L(n,\xi)}}\,. \tag{8}$$

The design algorithm is implemented in software Mathematica.

Software for designing elliptic filters

Some software tools already have functions for designing elliptic filters. Since we are going to use some modifications, we present the code in Mathematica [11] software as the best CAS software.

Firstly, we would like to show haw formulas are inputted into CAS software, and thus generate computer code.

The first cells contain filter specification. For example, we chose wideband low-pass filter and very sharp transition region. The pass-band edge frequency is 0.4 and the stopband edge frequency is 0.41 normalized to the sampling rate. The selectivity factor can be accurately computed if frequencies are specified as rational numbers, while the approximate number with 15 accurate decimals are obtained using the command N:

```
Fp = 4/10;
Fs = 41/100;
ξ = Tan[π Fs/2] / Tan[π Fp/2];
% // N

1.03342
```

The bilinear transformation is defined according to (2):

```
BT[s_, fp_] := (1 + s Tan[π fp]) / (1 - s Tan[π fp]);
BT[1, Fp] // N
```
```
-1.96261
```

The last value, -1.96261, is computed for passband frequency. Transfer function zeros for odd order filters are defined according to (3), and the first zero is negative:

```
X[n_Integer, ξ_, i_Integer] :=
  -JacobiCD[(2 i - 1)/n EllipticK[1/ξ^2], 1/ξ^2];
X[n_Integer, ξ_, i_Integer] := 0 /; And[i == (n + 1)/2, OddQ[n]];
X[5, ξ, 1] // N // Re
```
```
-0.989095
```

All zeros can be obtain for all indices, and the middle transfer function zero is 0:

```
X[n_Integer, ξ_] := X[n, ξ, #] & /@ Range[n];
X[5, ξ] // N // Re
```
```
{-0.989095, -0.813122, 0., 0.813122, 0.989095}
```

Discrimination factor can be computed using the definition (4), and the numeric value for the fifth-order filter and the specified selectivity factor is 22.304:

```
L[n_Integer, ξ_] := Module[{i},
   1/ξ^(n-2) (Product[(ξ^2 - X[n, ξ, i]^2)^2, {i, 1, (n-1)/2}] /
              Product[(1 - X[n, ξ, i]^2)^2, {i, 1, (n-1)/2}])];
L5mq = L[5, ξ] // N // Re
```
```
22.304
```

Auxiliary function is defined by (5), and the numeric value of EMQF filter is 0.954349:

```
Z[n_Integer, ξ_, ε_] :=
  JacobiSN[InverseJacobiSN[1/Sqrt[1 + ε^2], 1 - 1/(L[n, ξ])^2]
     EllipticK[1 - 1/ξ^2] / EllipticK[1 - 1/(L[n, ξ])^2], 1 - 1/ξ^2];
Z[5, ξ, 1/L5mq]
```
```
0.954349
```

The transfer function zeros can be computed using (6) and all poles must be in the left half-plane of the *s* domain.

```
S[n_Integer, ξ_, ε_, i_Integer] := Module[
   {x, ξ},
   x = X[n, ξ, i];
   ξ = Z[n, ξ, ε];
   (-ξ Sqrt[1 - ξ^2] Sqrt[1 - x^2] Sqrt[1 - x^2/ξ^2] + I x Sqrt[1 - (1 - 1/ξ^2) ξ^2]) /
    (1 - (1 - x^2/ξ^2) ξ^2)];
S[5, ξ, 1/L5mq, 1]
```
```
-0.0131701 - 1.03948 i
```

The poles are functions of the ripple factor, and the ripple factor is chosen according to the EMQF filter.

The transfer function of a digital filter uses the bilinear transformation of the transfer function zeros and poles.

```
Hz[n_, ξ_, ε_, Fp_, z_] := Module[{i, g, t},
   g =
    2
     (Product[((1 - 2 Re[BT[I ξ/X[n, ξ, i], Fp]] + 1) /
        (1 - 2 Re[BT[S[n, ξ, ε, i], Fp]] +
          Abs[BT[S[n, ξ, ε, i], Fp]]^2)), {i, 1, (n-1)/2}]) /
     (1 - BT[S[n, ξ, ε, (n + 1)/2], Fp]);
   t =
    (z + 1)/g
     (Product[((z^2 - 2 z Re[BT[I ξ/X[n, ξ, i], Fp]] + 1) /
        (z^2 - 2 z Re[BT[S[n, ξ, ε, i], Fp]] +
          Abs[BT[S[n, ξ, ε, i], Fp]]^2)), {i, 1, (n-1)/2}]) /
     (z - BT[S[n, ξ, ε, (n + 1)/2], Fp]);
   t];
```

The transfer function of the fifth-order EMQF filter for specified selectivity factor becomes:

```
H5 = Hz[5, N[ξ], 1/Sqrt[L5mq], Fp, z]
```
```
(0.638578 (1 + z) (1 + 1.64727 z + z^2) (1 + 1.7546 z + z^2)) /
 ((0.515584 + z) (0.774517 + 1.44556 z + z^2)
   (0.974893 + 1.60879 z + z^2))
```

In CAS, the accuracy can be set to the required number of digits. Fig. 1 shows magnitude response of the EMQF filter that has minimum attenuation in the stopband of 13.6 dB.

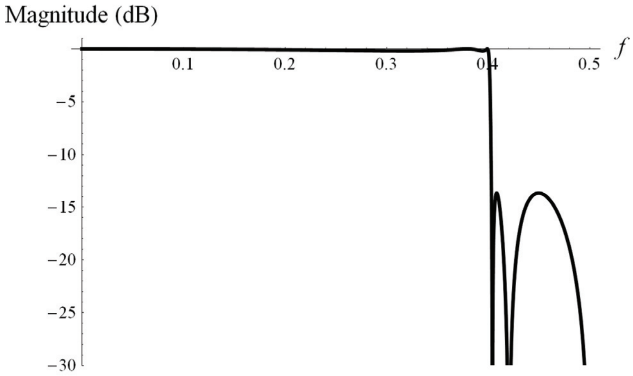

Fig. 1. Magnitude response of EMQF filter

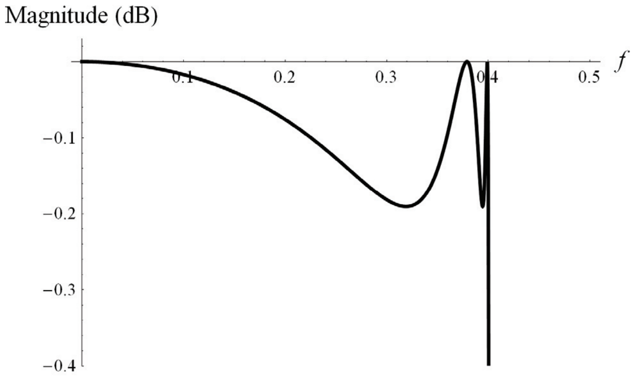

Fig. 2. Magnitude response of EMQF filter in passband

The passband ripple is less than 0.2 dB (Fig. 2).

The stopband attenuation can be increased *m* times in dB by the cascade connection of *m*=2, 3, …, 10 EMQF filters to more than 130 dB. Unfortunately, the passband ripple is *m* times larger, and for 130 dB minimum stopband attenuation, the passband ripple amount 2 dB.

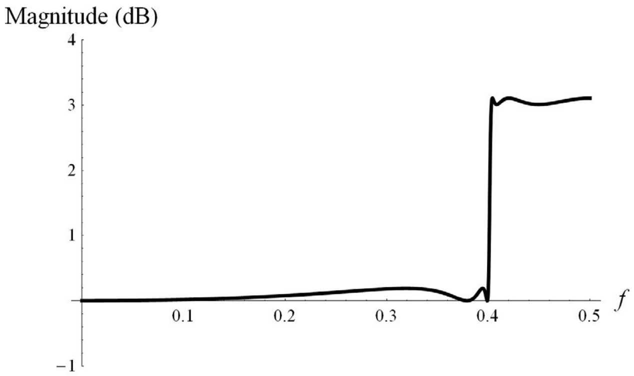

Fig. 3. Magnitude response of compensating section for *m*=2

Compensating section designed to improve the overall filter performance in the passband will be presented in the next section. Fig. 3 presents the magnitude response of the compensating section for *m*=2. The stopband attenuation will be reduced for about 3 dB.

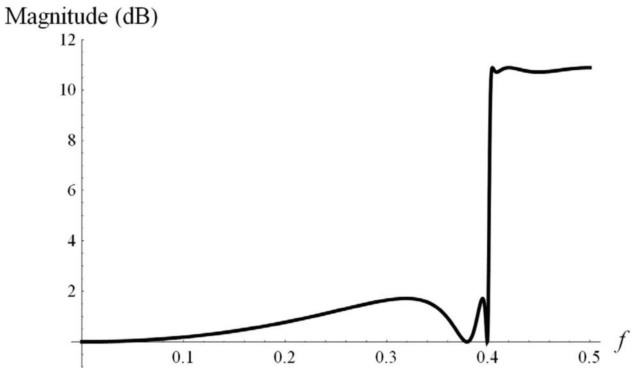

Fig. 4. Magnitude response of compensating section for *m*=10

Fig. 4 presents the magnitude response of the compensating section for *m*=10. The stopband attenuation will be reduced for about 11 dB.

For *m*=2, the filter consists of an EMQF section and one compensating section, with more than 10 dB attenuation in the stopband (Fig. 5), but with very small passband ripple of 0.002 dB (Fig. 6). With 10 cascaded EMQF sections and compensating section, the passband ripple will be still too small, 0.02 dB, and with more than 100 dB stopband attenuation.

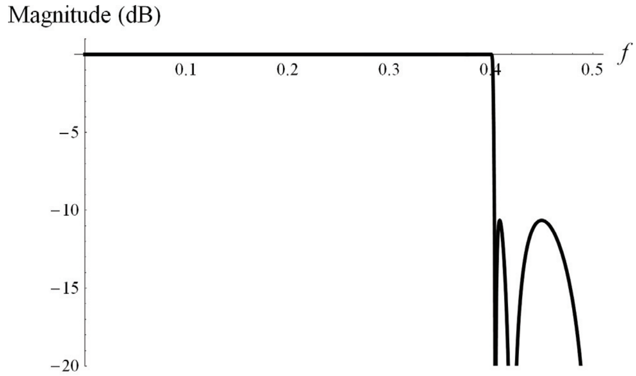

Fig. 5. Magnitude response for *m*=2 and compensating section

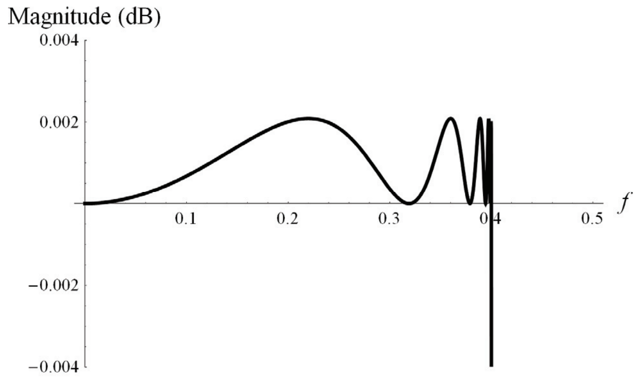

Fig. 6. Magnitude response for *m*=2 and compensating section

With 10 cascaded EMQF sections and only one compensating section (shown in Fig. 4), the minimum stopband attenuation is larger than 100 dB (Fig. 7) and still exhibits a small passband ripple of 0.1 dB (Fig. 8).

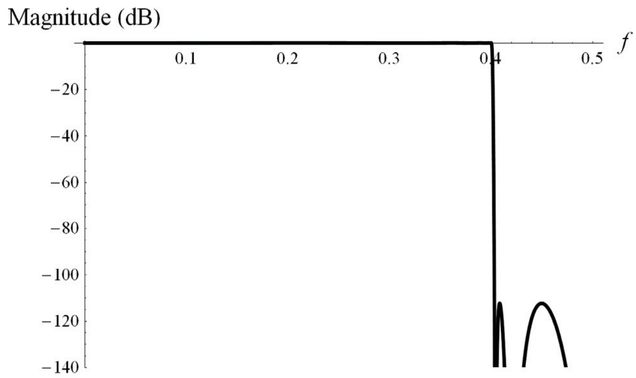

Fig. 7. Magnitude response for *m*=10 and one compensating section

It should be noticed that the fifth order EMQF filter can be implemented using all-pass branches with only 5 coefficients. Also, the compensating section can be implemented using the same filter structure for EMQF filters but with only 5 additional coefficients in the non-recursive path.

This means that with only 10 coefficients, a wide range of sharp filters can be implemented with small passband ripple and the minimum stopband attenuation from the range of 10 to 110 dB.

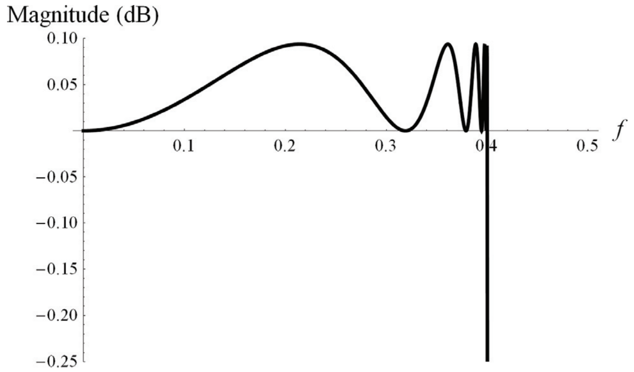

Fig. 8. Magnitude response in the passband for m=10 and one compensating section

The classic filter realization will require more than 50 coefficients to fulfill the same specifications.

Compensating section

The design of compensating section starts with the design of elliptic filter, but with the ripple factor different from that of EMQF filters. i.e.

$$\varepsilon_{CS} = \sqrt{\left(1+\frac{1}{L(n,\xi)}\right)^m - 1}\ . \qquad (9)$$

The poles of the elliptic filter with ripple factor computed using (9) become zeros of the compensating section, while poles of the compensating section are the same as the poles of EMQF filter.

Therefore, the zeros of the compensating section are within the unit circle, and compensating section is equiripple in the passband and in the stopband. The magnitude response with (m-1) EMQF sections and one compensating section (Fig. 9).

Fig. 10 presents the passband ripple for (m-1) EMQF sections and one compensating section.

The equivalent ripple of (m-1) EMQF sections and one compensating section can be computed using closed form expression

$$\varepsilon_r = \sqrt{\frac{\left((1+\varepsilon_{EMQF}^2)^m - 1\right)^m}{m^m \varepsilon_{EMQF}^2} \times \left(\frac{m-1}{(1+\varepsilon_{EMQF}^2)^m - \varepsilon_{EMQF}^2 - 1}\right)^{m-1} - 1}\ . \qquad (10)$$

The minimal value of the resulting filter of (m-1) EMQF sections and one compensating section can be computed as follows

$$L_r = \frac{1+\left(\left(1+\frac{1}{L(n,\xi)}\right)^m - 1\right)L(n,\xi)^2}{(1+L(n,\xi))^m}\ . \qquad (11)$$

The comparison of several EMQF filters and one compensating section or without compensating section is presented in Table 1.

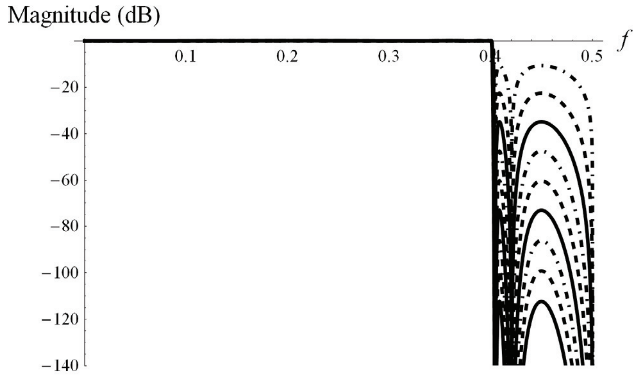

Fig. 9. Magnitude response for $m \in \{2,3,4,5,6,7,8,9,10\}$ and one compensating section

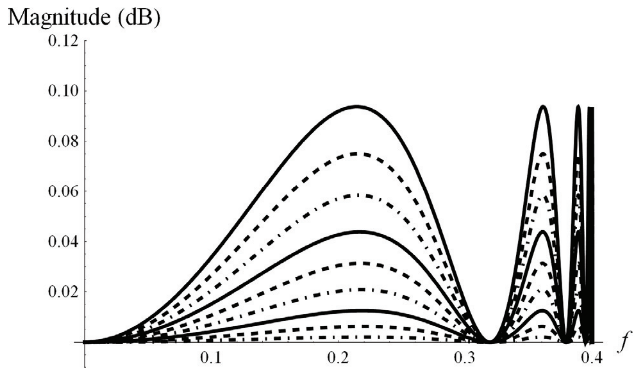

Fig. 10. Magnitude response in the passband for $m \in \{2,3,4,5,6,7,8,9,10\}$ and one compensating section

Table 1. Comparison of attenuations

EMQF × m	CS	Ripple dB	As dB	EMQF × m	CS	Ripple dB	As dB
1	0	0.19	13	1	1	0.002	10.6
2	0	0.38	27	2	1	0.006	22.5
3	0	0.57	41	3	1	0.01	34.8
4	0	0.76	54	4	1	0.02	47.4
5	0	0.95	68	5	1	0.03	60.2
6	0	1.14	82	6	1	0.04	73.1
7	0	1.33	95	7	1	0.06	86.1
8	0	1.52	109	8	1	0.08	99.2
9	0	1.71	123	9	1	0.10	112.3
10	0	1.90	136	10	1	0.11	125.5

Note: Cascaded 0 or 1 compensationg section and several EMQF filters. As is minimum stopband attenuation. Ripple is maximum pass-band attenuation.

From Table 1, it follows that the passband ripple of 10 EMQF filters and one compensating section is still lower than the ripple of only one EMQF filter.

Multirate implementation

The digital programmable devices such as FPGA can be used for implementing only one EMQF fifth-order filter with additional five coefficients for non-recursive branches. For repeated usage of the EMQF filter, we should use additional registers for storing intermediate results. The first input sample into EMQF filter is original sequence; the next input sample is from the output of EMQF filter. After filtering the sequence twice using EMQF filter, the sample is inputted again into EMQF filter. Processing the sequence m times using the same

EMQF filter exhibits the frequency response presented in Fig. 9 and Fig. 10.

This type of implementation is called multirate implementation because the processing rate of the filter is increased *m* times. The final result is obtained as a sum of non-recursive branches that implements the compensating section. This signal is down-sampled *m* times, and the rate of the filtered signal is again the same as the rate of the original signal.

Since we are using a low-order basic filter that occupies a small part of the programmable digital circuit, and the multirate technique for processing at highest rate, we call this type of filters efficient multirate filters.

Conclusion

The method for implementing sharp digital filter with a small passband ripple and large stopband attenuation, by making use of the fifth order elliptic filter with the minimal number of coefficients (only 5) is presented.

The same method can be used for variable cut-off frequency, using the very basic frequency transformation: the low-pass to low-pass filter transformation. The efficiency is in that the small amount of digital circuit is used for implementing filter coefficients and processing at highest rate than the rate of the original signal.

Acknowledgement

This work was partially supported by the Ministry of Education and Science of Serbia under Grant TR-32023.

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Received 2010 10 19
Accepted after revision 2011 07 27

M. Lutovac, V. Pavlovic. Efficient Multirate EMQF Digital Filters // Electronics and Electrical Engineering. – Kaunas: Technologija, 2012. – No. 3(119). – P. 39–44.

This work gives an original method that combines multirate technique and the most efficient elliptic filter of low order to implement sharp digital filters with minimum stopband attenuation from 10 do 100 dB, but with vary small pass-band ripple. The theory and the design procedure are presented in software Mathematica. Ill. 10, bibl. 11, tabl. 1 (in English; abstracts in English and Lithuanian).

M. Lutovac, V. Pavlovic. Daugiapakopių EMQF skaitmeninių filtrų efektyvumas // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2012. – Nr. 3(119). – P. 39–44.

Pateikiami daugiapakopių pirmųjų eilių eliptinių filtrų jungimo būdai, didinantys skaitmeninių filtrų efektyvumą esant minimaliam užtvarinės juostos silpninimui nuo 10 iki 100 dB. Atliktas projektavimas pateiktas matematinėje programoje „Mathematica“. Il. 10, bibl. 11, lent. 1 (anglų kalba; santraukos anglų ir lietuvių k.).