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Applications of Direct Lyapunov Method in Caputo Non-Integer Order Systems

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***Abstract*—Paper presents certain properties of Lyapunov direct method for non-integer order systems. Mittag-Leffler stability is defined and its relationship with Lyapunov stability is investigated. General results for Lyapunov functions are presented and a new result allowing constructive stability analysis is proved. Results are illustrated with the examples of stability analysis for problem of cooling, chemical reaction evolution and parametric stability analysis.**

***Index Terms*—Command and control systems, stability, stability criteria, Lyapunov methods, fractional calculus.**

I. INTRODUCTION

Non-integer order systems (often called fractional systems) are a rapidly developing field in technical and mathematical sciences. Most focus is oriented on their properties (see for example [1], [2]) and applications (see for example [3]–[6]). The goal of this paper is to highlight one of the interesting results from the first group.

Lyapunov direct method provides a way to analyse the stability of dynamical systems without solving the differential equations. It is especially advantageous when the solution is difficult or even impossible to find with classical methods. A basic analysis can be found in [7]–[9].

It is interesting to investigate an extension of the method for non-integer order systems. Such extension is based on the concept of Mittag-Leffler stability which is presented along with the appropriate theorem. Then we present some methods for finding the Lyapunov function for non-integer order systems.

II. PRELIMINARIES

Non-integer order calculus is important and rapidly developing field in modern control theory. In brief it is calculus with derivatives of non-integer order. In applications the most popular definitions of such derivatives are Caputo (1) and Riemann-Liouville (2) derivatives [10]:

$$ {}_{C}^{t_0}D_t^p f(t) = \frac{1}{\Gamma(n-p)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{p-n+1}}, \tag{1}$$

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$$ {}^{t_0}D_t^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} \frac{f(\tau)\, d\tau}{(t-\tau)^{p-n+1}}, \tag{2}$$

where $p \in (0,1)$, $n = \lceil p \rceil$ denotes the ceiling of p.

In analysis of non-integer order systems, the Mittag-Leffler function has key role. Two types of Mittag-Leffler function are used:

One-parameter Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{3}$$

where $\alpha > 0$ and $z \in C$.

Two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{4}$$

where $\alpha, \beta > 0$ and $z \in C$. It is customary to denote $E_{\alpha,1}(x) = E_\alpha(x)$. For $\alpha = 1$ and $\beta = 1$ we have $E_{1,1}(x) = e^x$. Therefore, it can be seen as a generalization of the exponential function. Mittag-Leffler function is used in the solution of such systems but also for stability analysis [7], [8].

III. NON-INTEGER ORDER SYSTEMS

In this paper we analyse the stability of Caputo systems [7], [8]. First let us consider a Caputo non-autonomous system (5)

$$ {}_{C}^{t_0}D_t^p x(t) = f(t,x), \tag{5}$$

with initial condition $x(t_0)$, where $p \in (0,1)$, $f:[t_0,\infty]\times\Omega \to R^n$ is piecewise continuous w.r.t. t, locally Lipschitz w.r.t. x on $[t_0,\infty]\times\Omega$ and Ω is a domain that contains the origin $x=0$. The equilibrium is defined in Definition 1.

Definition 1. The solution of Caputo system (5) such that $x(t) = x_0 = const$ is called the equilibrium.

Directly from the definition of Caputo derivative, we can see that x_0 is the equilibrium point if $f(t,x_0) = 0$, for $t > t_0$

[7], [8]. It can be also noted that it is sufficient to analyse only the equilibrium point at the origin. We can always transform the systems using substitution $y = x - x_0$. Similar analysis can be made for Riemann-Liouville derivative [7], [8].

It is possible to show the existence and uniqueness theorem for non-integer order differential equations [10], [11]. Similarly to integer-order systems, it is required that the function f is continuous and Lipschitz w.r.t. x. There are, however, some major differences, one of the most important being that there are two theorems: one for Caputo systems and one for Riemann-Liouville systems [11].

IV. MITTAG-LEFFLER STABILITY

Lyapunov stability theory is very important in nonlinear systems analysis of integer order. Fractional systems, however, have some unique properties which require a different approach. Therefore, so called Mittag-Leffler stability is introduced [7], [8].

Definition 2. (Mittag-Leffler stability) The solution of

$$ {}_{C}^{t_0}D_t^p x(t) = f(t,x), \tag{6}$$

is Mittag-Leffler stable if

$$ \|x(t)\| \le \left(m\left(x(t_0)\right) E_{\alpha}\left(-\lambda (t-t_0)^{\alpha}\right)\right) , \tag{7}$$

where t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \ge 0$, $\beta > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz for $x \in B \subset R^n$ with Lipschitz constant m_0.

In further analysis we will assume $t_0 = 0$ and omit it in the derivative symbol ${}_{C}^{t_0}D_t^p x(t) := {}_{C}D_t^p x(t)$.

In [7], [8] the authors claimed that asymptotic stability is direct a consequence of Mittag-Leffler stability, but they did not present any proof of this statement.

Proposition 1. Mittag-Leffler stability implies asymptotic stability.

Proof. We want to show that for every ε, there is δ, such that for every $x(0) < \delta$, we have $\|x(t)\| < \varepsilon$.

We have

$$ \|x(t)\| \le \left(m\left(x(0)\right) E_{\alpha}\left(-\lambda t^{\alpha}\right)\right) . \tag{8}$$

For $\alpha \in (0,1)$ and $t > 0$ we have

$$ E_{\alpha}\left(-\lambda t^{\alpha}\right) \le 1. \tag{9}$$

Hence

$$ m\left(x(0)\right)\left(E_{\alpha}\left(-\lambda t^{\alpha}\right)\right) \le m\left(x(0)\right). \tag{10}$$

Therefore

$$ \|x(t)\| \le m\left(x(0)\right). \tag{11}$$

Let Ω be a compact ball $\Omega_r = \left\{x \in R^n : \|x\| \le r\right\}$ where r is a given radius. Given that m is continuous and defined on R^n we have that $m(\Omega_r)$ is bounded (extreme value theorem) and $m^{\beta}(x)$ achieves its maximal and minimal values. Let $f(.)$ be a function $f: r \to f(r) = max\, m^{\beta}(x)$ on $\Omega_r \in R^n$. Function f is continuous because $m(\ )$ is locally Lipschitz. Let us take $\varepsilon = f(r)$. We have that for every $\|x_0\| < r$, the solution

$$ \|x(t)\| < m(x_0) < max\, m(\Omega_r) = \varepsilon. \tag{12}$$

Delta is smaller than the smallest solution of $\varepsilon = f(r)$. The solution exists because $f(r)$ is continuous and takes values from [0, ε]. Therefore, the origin is stable.

To prove asymptotic stability, it is sufficient to show the attractivity of the origin. It can be done directly from the definition of Mittag-Leffler stability. We have

$$ \|x(t)\| \le \left(m\left(x(0)\right) E_{\alpha}\left(-\lambda t^{\alpha}\right)\right) , \tag{13}$$

where $m(x_0)$ has a finite value and $E_{\alpha}\left(-\lambda t^{\alpha}\right) \to 0$ for $t \to \infty$ [12]. The solution $\|x(t)\|$ is bounded from above by a function convergent to zero and from below by zero. Hence, $\lim_{t\to\infty} \|x(t)\| = 0$. Therefore, the origin is asymptotically stable.

V. DIRECT LYAPUNOV METHOD FOR NON-INTEGER ORDER SYSTEMS

In this section, we will present an extension of Lyapunov direct method for non-integer order systems. This method can be used to verify Mittag-Leffler stability of Caputo systems (Theorem 2).

Theorem 2. Let x = 0 be an equilibrium point for the system

$$ {}_{C}^{t_0}D_t^p x(t) = f(t,x), \tag{14}$$

and $D \subset R^n$ be a domain containing the origin. Let $V(t,x(t)) : [0,\infty) \to D \subset R$ be a continuously differentiable function and locally Lipschitz with respect to x such that:

$$ \alpha_1 \|x\|^{a} \le V(t,x(t)) \le \alpha_2 \|x\|^{ab}, \tag{15}$$

$$ {}_{C}D_t V(t,x(t)) \le -\alpha_3 \|x\|^{ab}, \tag{16}$$

where $t \ge 0$, $x \in D$, β ∈ (0, 1), α_1, α_2, α_3, a, b are some positive constants. Then the norm of the solution is bounded by a Mittag-Leffler function

$$ \|x(t)\| \le \left(m\left(x(0)\right) E\left(\frac{-\alpha_3}{\alpha_2} t\right)\right)^{\frac{1}{a}}, \tag{17}$$

where m is a locally Lipschitz function.

If $\beta = p$ then the origin is Mittag-Leffler stable.

Proof. See [7], [8].

It is possible to relax the assumptions of Theorem 1 in order to verify asymptotic stability. The following approach uses class-K functions.

Definition 3. (class-K functions). A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to class-K if it is strictly increasing and $\alpha(0) = 0$ [13].

Theorem 3. Let $x = 0$ be an equilibrium point for the non-autonomous non-integer order Caputo system. Let us assume that there exists a Lyapunov function $V(t, x(t))$ and class-K functions α_i, $i = 1, 2, 3$ satisfying:

$$\alpha_1(\|x\|) \le V(t, x(t)) \le \alpha_2(\|x\|), \tag{18}$$

$$_C D_t \; V(t, x(t)) \le -\alpha_3(\|x\|), \tag{19}$$

where β $(0, 1)$. Then the Caputo system is asymptotically stable.

Proof. See [7], [8].

It can be observed, see for example [7]–[9], that direct application of theorems 2 and 3 is not very useful for checking the stability of non-integer order systems. There are, however, certain rules which allow constructive use of these theorems.

Theorem 4. Let $x(t) \in R$ be a continuous and differentiable function. Then for t greater or equal than 0, we have

$$\frac{1}{2}\, {}_C D_t^{\alpha}\, x^2(t) \le x(t)\, {}_C D_t^{\alpha}\, x(t), \tag{20}$$

where $\alpha \in (0, 1)$.

Proof. See [14].

Using this result, we can prove the following proposition:

Proposition 5. Caputo system

$${}^{t_0}_{C} D_t^{\alpha}\, x(t) = f(x), \tag{21}$$

is asymptotically stable if $x^{2^k - 1} f(x) < 0$ for a certain $k > 0$.

Proof. Let $V(x) = x^{2^k}$ be a Lyapunov candidate for the system. We have

$$\begin{aligned} {}_C D_t^{\alpha} V(x) &= {}_C D_t^{\alpha} \frac{1}{2} x^{2^k} = \frac{1}{2}\, {}_C D_t^{\alpha} x^{2^k} \le \\ &\le x^{\frac{2^k}{2}}\, {}_C D_t^{\alpha}\, x^{\frac{2^k}{2}} = x^{2^{k-1}}\, {}_C D_t^{\alpha}\, x^{2^{k-1}} < 0. \end{aligned} \tag{22}$$

Following this step k times we have

$$x^{\frac{2^k}{2} + \frac{2^k}{4} + \ldots + \frac{2^k}{2^k}} \times {}^{t_0}_{C} D_t^{\alpha}\, x(t) = x^{\frac{2^k}{2} + \frac{2^k}{4} + \ldots + \frac{2^k}{2^k}} \times f(x). \tag{23}$$

Now

$$\frac{2^k}{2} + \frac{2^k}{4} + \ldots + \frac{2^k}{2^k} = 2^k \sum_{i=1}^{k} \frac{1}{2^i} = 2^k - 1. \tag{24}$$

Hence, if $x^{\frac{2^k}{2} + \frac{2^k}{4} + \ldots + \frac{2^k}{2^k}} \times f(x) < 0$, then the system is asymptotically stable and Lyapunov function for this system can be $V(x) = x^{2^k}$.

VI. EXAMPLES

In the last section we will present three examples of stability analysis in non-integer order systems.

Example 1. Cooling of an iron bar [15]

First, let us consider the iron bar of length 1 m with specific heat $c = 0.4375 \frac{J}{gK}$, density $\rho = 7.88 \frac{g}{cm^3}$ and thermal conductivity $k = 0.836 \frac{W}{cmK}$ subject to initial condition $u_0 = 0$ [15].

The equation for cooling the bar with the above mentioned parameters (25)

$${}_C D_t^{1/2} x(t) = -\lambda x^4(t), \tag{25}$$

where $x(0) = 0$ and $\lambda = 0.24277$. The analysis of numerical solution can be found eg. in [15]. We will show the asymptotic stability of this system. Let $V(x) = \frac{1}{2} x^2$ be a Lyapunov candidate for system. Then

$$\begin{aligned} {}_C D_t^{1/2} V(x) &= {}_C D_t^{1/2} \frac{1}{2} x^2 = \frac{1}{2}\, {}_C D_t^{1/2} x^2 \\ &< x\, {}_C D_t^{1/2} x(t) < -x\lambda \cdot x^4 = -\lambda x^5. \end{aligned} \tag{26}$$

Assuming $x > 0$, we have $-\lambda x^5 \le 0$. The assumption is valid because $x = u_0 - u(0,t)$, where u denotes the temperature of the bar and in case of cooling $u(0,t) \le u_0$ which implies $x > 0$. Therefore, the system is asymptotically stable.

Example 2. Chemical reactions

Let us analyse a chemical reaction of three substrates

$$A + B + C \to products, \tag{27}$$

where the initial concentration is the same for every substrate $a = b = d$. Let c denote the instantaneous concentration of any substrate. Then the kinetic equation has the form (28)

$$\frac{dc}{dt} = -kc^3, \tag{28}$$

where k is a given parameter.

There are certain works which propose using non-integer order calculus for describing chemical processes. The following equation is proposed

$$ {}_C D_t^{\alpha} c(t) = -kc^3(t). \tag{29}$$

Let us take $V(c) = \frac{1}{2}c^2$. Then

$$ {}_C D_t^{\alpha} V(c) = {}_C D_t^{\alpha} \frac{1}{2}c^2 = \frac{1}{2} {}_C D_t^{\alpha} c^2 < c\, {}_C D_t^{\alpha} c(t) < $$
$$ < -ck \times c^3 = -kc^4. \tag{30}$$

Assuming that $k > 0$, the system is globally asymptotically stable.

Example 3. System with parameter

Let us take a system (31)

$$ {}_C D_t^{\alpha} x(t) = \sin x + kx. \tag{31}$$

The main goal is to find the values of parameter k such that the system described with (31) is asymptotically stable. Let us take $V(x) = \frac{1}{2}x^2$. Then

$$ {}_C D_t^{\alpha} V(x) = {}_C D_t^{\alpha} \frac{1}{2}x^2 = \frac{1}{2} {}_C D_t^{\alpha} x^2 < x\, {}_C D_t^{\alpha} x(t) = $$
$$ = x \times \sin x + kx^2 \approx x^2 + kx^2 = (k+1)x^2. \tag{32}$$

The last two equalities are made under assumption that $\sin x = x$ if $x \in (-\pi/36, \pi/36)$. Assuming that x is in the given interval, we have that for $k < -1$, it is true that $(k+1)x^2 < 0$. Hence, the origin is locally asymptotically stable for $k < -1$.

VII. CONCLUSIONS

Presented extension of direct Lyapunov method for non-integer order systems is a part of ongoing research. One can easily extend the class of Lyapunov functions used in proof of Proposition 5 to their linear combinations. It is, however, interesting to find results similar to Theorem 4 which will work for more general functions, such as quadratic forms. Existence of similar conditions is currently an open question. It should be however noted, that no constructive results in Lyapunov functions for systems of non-integer orders were available till 2014 ([14], [16]), so the field develops rapidly.

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