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A new Approach to Calculate the Current Density and the Surface Charge Density in the Conductor Carrying Steady State Current, employing the Gradient of the Scalar Function-resistance

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Introduction

It is an easy task to calculate the resistance of a conductor carrying steady state current. In order to find the current density in the same conductor, the task becomes harder for especially complicated geometries which may require a solution of the Laplace's equation [1], $\nabla^2 V = 0$ then the gradient of the potential function, $\vec{\mathbf{J}} = -\sigma\nabla V$, where $\vec{\mathbf{J}}$ – current density, V – electric potential. The application of this method which is widely in use is easy for some geometries like CaCS (Cartesian coordinate system) and CyCS (Cylindrical coordinate system) but requires special attention when other geometries like the SpCS (Spherical coordinate system), OSCS (Oblate spheroidal coordinate system), PSCS (Prolate spheroidal coordinate system), TOCS (Toroidal coordinate system), BSCS (Bispherical coordinate system), etc. [2,3] are in consideration. In this study, a straightforward calculation of the current density in the conductor for the CyCS and the OSCS is performed by using the equation, $\vec{\mathbf{J}} = -\sigma I\nabla R$. The calculations also revealed the source of the surface charge density on the surface of the conductor carrying steady state current. The calculations are given in the following sections.

Properties of the CyCS and the OSCS

Properties of the CyCS, ρ, ϕ, z :

Limits of the variables:

$$0 \le \rho \le \infty,\ 0 \le \phi \le 2\pi,\ -\infty \le z \le \infty \tag{1}$$

Differential length elements:

$$\overrightarrow{d\ell}_\rho = d\rho\,\vec{\mathbf{a}}_\rho, \qquad \overrightarrow{d\ell}_\phi = \rho\,d\phi\,\vec{\mathbf{a}}_\phi, \qquad \overrightarrow{d\ell}_z = dz\,\vec{\mathbf{a}}_z \tag{2}$$

Gradient operator in the CyCS:

$$\nabla = \frac{\partial}{\partial\rho}\vec{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\vec{\mathbf{a}}_\phi + \frac{\partial}{\partial z}\vec{\mathbf{a}}_z \tag{3}$$

Properties of the OSCS, λ, μ, ϕ :

Limits of the variables:

$$-\infty \le \lambda \le \infty,\ -1 \le \mu \le 1,\ 0 \le \phi \le 2\pi \tag{4}$$

Differential length elements:

$$\overrightarrow{d\ell}_\lambda = a\sqrt{\frac{\lambda^2+\mu^2}{1+\lambda^2}}d\lambda\,\vec{\mathbf{a}}_\lambda, \quad \overrightarrow{d\ell}_\mu = a\sqrt{\frac{\lambda^2+\mu^2}{1-\mu^2}}d\mu\,\vec{\mathbf{a}}_\mu$$

$$\overrightarrow{d\ell}_\phi = a\sqrt{(1+\lambda^2)(1-\mu^2)}d\phi\,\vec{\mathbf{a}}_\phi, \tag{5}$$

where μ – permeability.

Gradient operator in the OSCS:

$$\nabla = \left(\begin{array}{l} \dfrac{\sqrt{1+\lambda^2}}{a\sqrt{\lambda^2+\mu^2}}\dfrac{\partial}{\partial\lambda}\vec{\mathbf{a}}_\lambda \\ +\dfrac{\sqrt{1-\mu^2}}{a\sqrt{\lambda^2+\mu^2}}\dfrac{\partial}{\partial\mu}\vec{\mathbf{a}}_\mu \\ +\dfrac{1}{a\sqrt{(1+\lambda_2)(1-\mu^2)}}\dfrac{\partial}{\partial\phi}\vec{\mathbf{a}}_\phi \end{array} \right) \tag{6}$$

Ohm's Law and the gradient of the scalar function-resistance

The well known Ohm's law is described as

$$R = \frac{\ell}{\sigma s} = \frac{\int_{\ell_1}^{\ell_2}\vec{d\ell}}{\sigma\int_{s_1}^{s_2}\vec{ds}}, \tag{7}$$

where R – electric resistance, σ – electric conductivity.

When a medium described by the differential volume element, $dv = \vec{d\ell}\cdot\vec{ds}$ is filled with an isotropic conducting material (σ), in the direction of $\vec{d\ell}$, the exact parametric resistances in the orthogonal directions can easily be found as below:

resistance in ρ direction:

$$R_\rho = \frac{\int \vec{d\ell}_\rho}{\sigma \int \vec{d\ell}_\phi \times \int \vec{d\ell}_z} = \frac{\int_{\rho_1}^{\rho_2} d\rho}{\sigma \int_{\phi_1}^{\phi_2} \rho d\phi \int_{z_1}^{z_2} dz}, \qquad (8)$$

$$R_\rho = \frac{\ln(\rho_2/\rho_1)}{\sigma(\phi_2-\phi_1)(z_2-z_1)}; \qquad (9)$$

resistance in ϕ direction:

$$R_\phi = \frac{\int \vec{d\ell}_\phi}{\sigma \int \vec{d\ell}_\rho \times \int \vec{d\ell}_z} = \frac{\int_{\phi_1}^{\phi_2} \rho d\phi}{\sigma \int_{\rho_1}^{\rho_2} d\rho \int_{z_1}^{z_2} dz}, \qquad (10)$$

$$R_\phi = \frac{(\phi_2-\phi_1)}{\sigma \ln(\rho_2/\rho_1)(z_2-z_1)}; \qquad (11)$$

resistance in z direction:

$$R_z = \frac{\int_{z_1}^{z_2} dz}{\sigma \int_{\rho_1}^{\rho_2} d\rho \times \int_{\phi_1}^{\phi_2} \rho d\phi} = \frac{2(z_2-z_1)}{\sigma(\rho_2^2-\rho_1^2)(\phi_2-\phi_1)} \qquad (12)$$

for the CyCS since the current may be confined to a region by insulating boundaries, without causing fringing fields [4]. In eq. (7–12) $\rho_2 > \rho_1$, $\phi_2 > \phi_1$ and $z_2 > z_1$. The electric field intensity in any direction (eg. ρ, ϕ and z) may not easily be calculated since it may require the solution of the Laplace's equation, $\nabla^2 V = 0$ then the gradient of the potential function, $\vec{\mathbf{E}} = -\nabla V$, where $\vec{\mathbf{E}}$ – electric field intensity. The calculations become harder for other complicated geometries (e.g. the OSCS, PSCS, ToCS, BSCS, etc.).

A novel and easy method of calculating the electric field intensity without involving complicated calculations and any assumptions is presented. The suggested approach only uses $\vec{\mathbf{E}} = -I\nabla R$, where I – electric current. An application of $\vec{\mathbf{E}} = -I\nabla R$ (eq.1) in the orthogonal directions for the CyCS is shown below:

$$\vec{\mathbf{E}}_\rho = \frac{I}{\sigma}\begin{pmatrix} -\dfrac{1}{\rho_2(\phi_2-\phi_1)(z_2-z_1)}\vec{\mathbf{a}}_\rho \\ +\dfrac{\ln(\rho_2/\rho_1)}{\rho_2(\phi_2-\phi_1)^2(z_2-z_1)}\vec{\mathbf{a}}_\phi \\ +\dfrac{\ln(\rho_2/\rho_1)}{(\phi_2-\phi_1)(z_2-z_1)^2}\vec{\mathbf{a}}_z \end{pmatrix}, \qquad (13)$$

$$\vec{\mathbf{E}}_\phi = \frac{I}{\sigma}\begin{pmatrix} \dfrac{(\phi_2-\phi_1)}{\rho_2[\ln(\rho_2/\rho_1)]^2(z_2-z_1)}\vec{\mathbf{a}}_\rho \\ -\dfrac{1}{\rho_2\ln(\rho_2/\rho_1)(z_2-z_1)}\vec{\mathbf{a}}_\phi \\ +\dfrac{(\phi_2-\phi_1)}{\ln(\rho_2/\rho_1)(z_2-z_1)^2}\vec{\mathbf{a}}_z \end{pmatrix}, \qquad (14)$$

$$\vec{\mathbf{E}}_z = \frac{I}{\sigma}\begin{pmatrix} \dfrac{2(z_2-z_1)}{(\rho_2^2-\rho_1^2)^2(\phi_2-\phi_1)}\vec{\mathbf{a}}_\rho \\ +\dfrac{2(z_2-z_1)}{\rho_2(\rho_2^2-\rho_1^2)(\phi_2-\phi_1)^2}\vec{\mathbf{a}}_\phi \\ -\dfrac{2}{(\rho_2^2-\rho_1^2)(\phi_2-\phi_1)}\vec{\mathbf{a}}_z \end{pmatrix}. \qquad (15)$$

The gradient operation is performed over the second variables, ρ_2, ϕ_2, z_2 while the first variables, ρ_1, ϕ_1, z_1 are kept fixed. In eq.(13–15), there are 3 components of $\vec{\mathbf{E}}$ in any orthogonal directions which represent the current densities in the orthogonal directions to be obtained by using $\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$ and the surface charge densities [5–9] to be obtained by using $\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}$, where $\vec{\mathbf{D}}$ – electric flux density, ε – permittivity. For example, in eq.(9), the z component of $\vec{\mathbf{E}}_z$ represents the electric field in the conductor in the direction of current flow, while ρ and ϕ components of $\vec{\mathbf{E}}_z$ represent the source of surface charge densities on the conductor due to current in the conductor of finite size limited by $0 \le \rho_1 < \rho_2 < \infty$, $0 \le \phi_1 < \phi_2 < 2\pi$ and $0 \le z_1 < z_2 < \infty$.

The calculations for the OSCS are given as below as the further examples. The gradient operation (6) is again performed over the second variables, λ_2, μ_2, ϕ_2 while the first variables, λ_1, μ_1, ϕ_1 are kept fixed.

Resistance in λ direction:

$$R_\lambda = \frac{\int \vec{d\ell}_\lambda}{\sigma \int \vec{d\ell}_\mu \times \int \vec{d\ell}_\phi}, \qquad (16)$$

$$R_\lambda = \frac{\int_{\lambda_1}^{\lambda_2} a\sqrt{\dfrac{\lambda^2+\mu^2}{1+\lambda^2}}d\lambda}{\sigma \int_{\mu_1}^{\mu_2} a\sqrt{\dfrac{\lambda^2+\mu^2}{1-\mu^2}}d\mu \int_{\phi_1}^{\phi_2} a\sqrt{(1+\lambda^2)(1-\mu^2)}d\phi}, \qquad (17)$$

$$R_\lambda = \frac{\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1}{\sigma a(\mu_2-\mu_1)(\phi_2-\phi_1)}. \qquad (18)$$

Resistance in μ direction:

$$R_\mu = \frac{\int \vec{d\ell}_\mu}{\sigma \int \vec{d\ell}_\lambda \times \int \vec{d\ell}_\phi}, \tag{19}$$

$$R_\mu = \frac{\int_{\mu_1}^{\mu_2} a\sqrt{\frac{\lambda^2+\mu^2}{1-\mu^2}}d\mu}{\sigma\int_{\lambda_1}^{\lambda_2} a\sqrt{\frac{\lambda^2+\mu^2}{1+\lambda^2}}d\lambda \int_{\phi_1}^{\phi_2} a\sqrt{(1+\lambda^2)(1-\mu^2)}d\phi}, \tag{20}$$

$$R_\mu = \frac{\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)}}{2\sigma a(\lambda_2-\lambda_1)(\phi_2-\phi_1)}. \tag{21}$$

Resistance in ϕ direction:

$$R_\phi = \frac{\int \vec{d\ell}_\phi}{\sigma \int \vec{d\ell}_\lambda \times \int \vec{d\ell}_\mu}, \tag{22}$$

$$R_\phi = \frac{\int_{\phi_1}^{\phi_2} a\sqrt{(1+\lambda^2)(1-\mu^2)}d\phi}{\sigma\int_{\lambda_1}^{\lambda_2} a\sqrt{\frac{\lambda^2+\mu^2}{1+\lambda^2}}d\lambda \int_{\mu_1}^{\mu_2} a\sqrt{\frac{\lambda^2+\mu^2}{1-\mu^2}}d\mu}, \tag{23}$$

$$R_\phi = \frac{2(\phi_2-\phi_1)}{\sigma a\left((\lambda_2-\lambda_1)\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)} - 2(\mu_2-\mu_1)(\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1)\right)}. \tag{24}$$

Electric field intensities in the orthogonal directions:

$$\vec{\mathbf{E}}_\lambda = \frac{I}{\sigma}\left(\begin{array}{l} -\dfrac{1}{a^2\sqrt{\lambda_2^2+\mu_2^2}\sqrt{1+\lambda_2^2}(\mu_2-\mu_1)(\phi_2-\phi_1)}\vec{\mathbf{a}}_\lambda \\ +\dfrac{\sqrt{1-\mu_2^2}(\tan^{-1}\lambda_2-\tan^{-1}\lambda_1)}{a^2\sqrt{\lambda_2^2+\mu_2^2}(\mu_2-\mu_1)^2(\phi_2-\phi_1)}\vec{\mathbf{a}}_\mu \\ +\dfrac{(\tan^{-1}\lambda_2-\tan^{-1}\lambda_1)}{a^2\sqrt{(1+\lambda_2^2)(1-\mu_2^2)}(\mu_2-\mu_1)(\phi_2-\phi_1)^2}\vec{\mathbf{a}}_\phi \end{array}\right), \tag{25}$$

$$\vec{\mathbf{E}}_\mu = \frac{I}{\sigma}\left(\begin{array}{l} \dfrac{\sqrt{1+\lambda_2^2}\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)}}{2a^2\sqrt{\lambda_2^2+\mu_2^2}(\lambda_2-\lambda_1)^2(\phi_2-\phi_1)}\vec{\mathbf{a}}_\lambda \\ -\dfrac{1}{a^2\sqrt{\lambda_2^2+\mu_2^2}\sqrt{1-\mu_2^2}(\lambda_2-\lambda_1)(\phi_2-\phi_1)}\vec{\mathbf{a}}_\mu \\ +\dfrac{\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)}}{2a^2\sqrt{(1+\lambda_2^2)(1-\mu_2^2)}(\lambda_2-\lambda_1)(\phi_2-\phi_1)^2}\vec{\mathbf{a}}_\phi \end{array}\right), \tag{26}$$

$$\vec{\mathbf{E}}_\phi = \frac{I}{\sigma}\left(\begin{array}{l} \dfrac{2(\phi_2-\phi_1)\sqrt{1+\lambda_2^2}\left(\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)} - \frac{2(\mu_2-\mu_1)}{1+\lambda_2^2}\right)}{a^2\sqrt{\lambda_2^2+\mu_2^2}\left((\lambda_2-\lambda_1)\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)} - 2(\mu_2-\mu_1)(\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1)\right)^2}\vec{\mathbf{a}}_\lambda \\ +\dfrac{4(\phi_2-\phi_1)\sqrt{1-\mu_2^2}\left(\frac{\lambda_2-\lambda_1}{1-\mu_2^2} - 2(\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1)\right)}{a^2\sqrt{\lambda_2^2+\mu_2^2}\left((\lambda_2-\lambda_1)\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)} - 2(\mu_2-\mu_1)(\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1)\right)^2}\vec{\mathbf{a}}_\mu \\ -\dfrac{2}{a^2\sqrt{(1+\lambda_2^2)(1-\mu_2^2)}\left((\lambda_2-\lambda_1)\times\ln\frac{(\mu_2+1)(\mu_1-1)}{(\mu_2-1)(\mu_1+1)} - 2(\mu_2-\mu_1)(\tan^{-1}\lambda_2 - \tan^{-1}\lambda_1)\right)}\vec{\mathbf{a}}_\phi \end{array}\right). \tag{27}$$

In (16-27), $\lambda_2 > \lambda_1$, $\mu_2 > \mu_1$ and $\phi_2 > \phi_1$.

In λ direction, the resistance and the current density represent the spreading/constriction resistance and current density respectively. The mentioned quantities were obtained by the transformation from the capacity solutions and the charge density analogy (projection principle) previously [10–11].

Unit vector transformations between the OSCS and the CaCS,

$$\begin{bmatrix}\vec{\mathbf{a}}_\lambda \\ \vec{\mathbf{a}}_\mu \\ \vec{\mathbf{a}}_\phi\end{bmatrix} = \begin{bmatrix} \lambda\sqrt{\frac{1-\mu^2}{\lambda^2+\mu^2}}\cos\phi & \lambda\sqrt{\frac{1-\mu^2}{\lambda^2+\mu^2}}\sin\phi & \mu\sqrt{\frac{\lambda^2+1}{\lambda^2+\mu^2}} \\ -\mu\sqrt{\frac{\lambda^2+1}{\lambda^2+\mu^2}}\cos\phi & -\mu\sqrt{\frac{\lambda^2+1}{\lambda^2+\mu^2}}\sin\phi & \lambda\sqrt{\frac{1-\mu^2}{\lambda^2+\mu^2}} \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \times \begin{bmatrix}\vec{\mathbf{a}}_x \\ \vec{\mathbf{a}}_y \\ \vec{\mathbf{a}}_z\end{bmatrix} \tag{28}$$

gives a further insight into the vector properties of the current density for the OSCS. The radial (ρ) and axial (z) components of the fields, $\vec{\mathbf{J}}$ and $\vec{\mathbf{E}}$ can easily be obtained and from which radial and axial components of the resistance can be calculated by using the equation below.

$$P = I^2 R = \int_v p\,dv = \int_v \vec{\mathbf{J}}\cdot\vec{\mathbf{E}}\,dv, \tag{29}$$

where P – power, p – power density.

Conclusions

The gradient of the scalar function-resistance is used to find the electric field intensity and current density in the conductor carrying steady state current where traditionally the Laplace's equation is employed. The components of the gradient also give the source of the surface charge density due to current carrying conductor. The similar comments may be developed for electric and magnetic fields described by

$$\vec{\mathbf{E}} = -Q\nabla\frac{1}{C} \text{ and } \vec{\mathbf{H}} = -\Phi\nabla\Re \tag{30}$$

respectively; here C – capacitance, $\Re$ – reluctance, Φ – magnetic flux, Q – electric charge, $\vec{\mathbf{H}}$ – magnetic flux intensity. Here, the fringing fields must be taken into consideration due to the finite size effects.

References

1. **Hayt W. H.** Engineering Electromagnetics. – 5th ed. – MacGraw-Hill, NY, 1989.
2. **Moon P., Spencer D. E.** Field Theory Handbook. – 2nd ed. – Springer-Verlag, Berlin, 1988.
3. **Moon P., Spencer D. E.** Field Theory for Engineers. – Van Nostrand, Princeton, NJ, 1961.
4. **Smythe W. R.** Static and Dynamic Electricity. – Hemisphere Pub. – 3rd ed. – 1989.
5. **Matzek M. A., Russell B. R.** On the Transverse Electric Field within a Conductor Carrying a Steady Current // Am. J. Phys. – . – 1968. – Vol. 36. – P. 905.
6. **Assis A. K. T., Rodrigues Jr. W. A., Mania A. J.** The Electric Field Outside a Stationary Resistive Wire Carrying a Constant Current // Foundations of Physics. – 1999. – Vol.29, No. 5. –P. 729–753.
7. **Jackson J. D.** Surface charges on circuit wires and resistors play three roles // Am. J. Phys. – 1996. – Vol. 64/7. – P. 855–870.
8. **Preyer N. W.** Transient behavior of simple RC circuits // Am. J. Phys. – 2002. – Vol. 70. – P. 1187.
9. **Parker S.** Electrostatic and current flow // Am. J. Phys. – 1970. – Vol. 38/6. – P. 720.
10. **Holm R.** Electric Contacts, Theory and Applications. – Berlin, Springer-Verlag, 1976.
11. **Lindell I. V.** Charge density on a conducting ellipsoid and an elliptic disk // Am. J. Phys. – 1997. – Vol. 65. – P. 1113.

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It is shown that the gradient of the scalar function-resistance can be used to determine the electric field intensity, $\vec{\mathbf{E}} = -I\nabla R$ and the current density, $\vec{\mathbf{J}} = -\sigma I\nabla R$ in the conductor carrying steady state current. Application of the suggested method is straightforward and does not require heavy calculations such as the Laplace's equation which requires assumptions especially in the case of complicated geometries. The calculations also revealed the source of the surface charge density – obtained from the transversal component of the electric field intensity – on the surface of the conductor formed by the steady state current. Bibl. 11 (in English; summaries in English, Russian and Lithuanian).

О. Гурдал. Новый метод вычисления плотности электрического тока и поверхностной плотности заряда в проводнике, используя градиент функции скалярного сопротивления // Электроника и электротехника. – Каунас: Технология, 2008. – № 8(88). – С. 33-36.

Показано, что градиент функции скалярного сопротивления может использоваться для определения интенсивности электрического поля $\vec{\mathbf{E}} = -I\nabla R$ и плотности электрического тока в проводнике $\vec{\mathbf{J}} = -\sigma I\nabla R$. Предложенный метод отличается простым применением и не требует интенсивных вычислений. Вычисления также показали, что источник поверхностной плотности заряда –трансверсальный компонент интенсивности электрического поля. Библ. 11 (на английском языке; рефераты на английском, русском и литовском яз.).

O. Gürdal. Naujas skaliarine varžos funkcija pagrįstas metodas srovės ir paviršinio krūvio tankiui nuolatinės srovės laidininkuose apskaičiuoti // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2008. – Nr. 8(88). – P. 33–36.

Parodyta, jog skaliarinės varžos funkcijos gradientas gali būti naudojamas nuolatinės srovės laidininkų kuriamo elektrinio lauko intensyvumui $\vec{\mathbf{E}} = -I\nabla R$ ir srovės tankiui $\vec{\mathbf{J}} = -\sigma I\nabla R$ apskaičiuoti. Siūlomą metodą taikyti gana nesudėtinga, nereikia didelės apimties skaičiavimų. Skaičiavimai taip pat rodo, jog paviršinio krūvio šaltinis yra skersinio elektrinio lauko stiprio komponentas. Bibl. 11 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).