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Digital Signal Processing UD Method and its Statistical Characteristics

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Introduction

The range of literature on UD method is very wide. Number of publications on the matter reaches into hundreds. UD method was used for investigating various problems in biology, medicine, pharmacology, toxicology and signal processing [1, 2, 3, 4, 5].

This paper investigates UD method just mathematically and does not link it with any particular application. However the objective of the analysis is to obtain statistical characterisations of the UD procedure (method), taking in account of which may improve filtering interferences of the stroboscopic signal processing.

We consider kind of classical UD method that satisfies the following three conditions:

1) the step length s of UD procedure is the same in both directions;
2) the step length s remains constant at all stages;
3) the perturbing factor distribution is normal with zero mean.

Thus let us consider UD procedure determined by normally distributed random variable X with the mean $EX=0$ and variance $DX=\sigma_0^2$, the step $s=\Delta\sigma_0$, where $0.1\leq \Delta \leq 1$ and in start position starting point of the coordinate axis is $x_0=0$. At the first step the UD process gets in point $x_1=s$ with probability $P\{X>0\}$ and in point $x_2=-s$ with probability $P\{X\leq 0\}$. If the UD process was in point ks after n steps, then in the step $n+1$ it will get into point $(k+1)s$ with probability $P\{X>ks\}$ or in point $(k-1)s$ with probability $P\{X\leq ks\}$.

So, if the UD process has made exactly n steps, then the probability that process endpoint is ks depends on n and k parity. If n is even number and k is odd then this probability (we denote it $p_k(n,s)$) is equal to zero. If n is odd number and k is even then again $p_k(n,s)=0$. Therefore in the analysis we consider separately instances when $n=2m$ and $n=2m+1$. We designate random variable $X_n(s)$ representing state probability distribution after n-th step of the UD procedure.

Since

$$P\{X_n(s)=ks\}=p_k(n,s), \tag{1}$$

we have

$$\sum_{k=-n}^{k=n} p_k(n,s)=1 \tag{2}$$

$$p_k(n,s)=p_{-k}(n,s). \tag{3}$$

From the last it follows that

$$EX_n(s)=0,\ \ DX_n(s)=E(X_n(s))^2=2\sum_{k=1}^{n}k^2s^2p_k(n,s). \tag{4}$$

From description of UD procedure at step $n+1$ follows that mathematically the UD process is represented by numerable simple and homogeneous Markov chain over set of states $S=\{0, 1, -1, 2, -2, 3, -3, \ldots\}$ with transition probability matrix $\boldsymbol{M}_s=(a_{ij})$ and initial distribution $\pi_0=(1, 0, 0, \ldots)$, if the state numbering is done in the above shown sequence and elements a_{ij} are defined by equations:

$$\begin{cases} a_{12}=a_{13}=P\{X>0\}, \\ a_{21}=P\{X\leq s\},\ a_{24}=P\{X>s\}, \\ a_{31}=P\{X>-s\},\ a_{35}=P\{X\leq -s\}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{2k,\,2k-2}=P\{X\leq ks\},\ a_{2k,\,2k+2}=P\{X>ks\}, \\ a_{2k+1,\,2k-1}=P\{X>-ks\},\ a_{2k+1,\,2k+3}=P\{X\leq -ks\}, \\ k=2, 3, 4, \ldots \end{cases} \tag{5}$$

Since the random variable X is distributed according to normal distribution law,

$$P\{X>ks\}=P\{\frac{X}{\sigma_0}>k\Delta\}=P\{\frac{X}{\sigma_0}<-k\Delta\}=\Phi(-k\Delta), \tag{6}$$

where Φ – standard normal distribution function.

If the process of UD procedure after the n-th step has got in state ks, then the next two steps will bring the process to state $(k-2)s$ with probability

$$a_k=\Phi(k\Delta)\Phi((k-1)\Delta), \tag{7}$$

to state ks with probability

$$b_k=\Phi(k\Delta)\Phi(-(k-1)\Delta)+\Phi(-k\Delta)\Phi((k+1)\Delta) \tag{8}$$

and to state *(k+2)s* with probability

$$c_k=\Phi(-k\Delta)\Phi(-(k+1)\Delta). \tag{9}$$

Thus UD process, where number of steps is even number, is represented by simple homogeneous Markov chain over set of states $\breve{S}=\{0,\ 2,\ -2,\ 4,\ -4,\ldots\}$ with transition probability matrix $\breve{M}_s=(b_{ij})$, where state numbering was done in the above shown sequence and elements b_{ij} are defined by equations:

$$\begin{cases} b_{11}=\Phi(\Delta),\ b_{12}=\Phi(0)\Phi(-\Delta),\ b_{13}=\Phi(0)\Phi(-\Delta), \\ b_{21}=a_2,\ b_{22}=b_2,\ b_{24}=c_2, \\ b_{31}=a_2,\ b_{33}=b_2,\ b_{35}=c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{2k,2k-2}=a_{2k},\ b_{2k,2k}=b_{2k},\ b_{2k,2k+2}=c_{2k}, \\ b_{2k+1,2k-1}=a_{2k},\ b_{2k+1,2k+1}=b_{2k},\ b_{2k+1,2k+3}=c_{2k}, \\ k=2,3,4,\ldots \end{cases} \tag{10}$$

In compliance with $\breve{M}_s$ elements definitions we can write: $b_{14,16}=c_{14}$ and $b_{15,17}=c_{14}$. By dividing rows 14 and 15 of matrix $\breve{M}_s$ by $1-c_{14}$ we obtain matrix whose common elements of first 15 rows and columns represent regular stochastic matrix $M_s^{'}=(b_{ij}^{'})$. Matrix $M_s^{'}$ together with stochastic vector $\pi_0^{'}=(p_1^{'},p_2^{'},\ldots,p_{15}^{'})$, where $p_1^{'}=1$, represents finite Markov chain, which in several aspects substitutes the numerable Markov chain with transition probability matrix $\breve{M}_s$. If $s\geq 0.5\sigma_0$, then we can replace the numerable Markov chain by finite Markov chain over set of states {*0, 2, -2, 4, -4, 6, -6*}. In this case matrix $M_s^{'}$ will be replaced by matrix M_s^{*} embodying common elements of 7 rows and 7 columns of matrix $\breve{M}_s$ after dividing elements of rows 6 and 7 by $1-c_6$.

Raising the matrix $M_s^{'}$ (respectively M_s^{*}) to the *N*-th power we obtain matrix whose elements of first 11 rows and columns (respectively elements of the first three rows and columns) will coincide with the corresponding elements of matrix $(\breve{M}_s)^N$. Elements of twelfth and thirteenth rows, respectively columns, of matrix $(M_s^{'})^N$ will differ from the respective elements of matrix $(\breve{M}_s)^N$ by coefficient *1/(1-c*$_{16}$*)*. Since c_{16} is very small for all $\Delta\geq 1$, the elements of these rows and columns are practically the same as the respective elements of matrix $(\breve{M}_s)^N$. The same applies to elements of forth and fifth rows (respectively columns) of matrix $(M_s^{*})^N$ when $\Delta\geq 0.5$.

If *N* is sufficiently big, e.g. $N\geq 20$, then the rows of matrix $(M_s^{'})^N$ and matrix $(M_s^{*})^N$ will be practically equal, and elements of last two columns can be calculated using the fact that the row vectors are stochastic vectors. This circumstance allows for considering the UD process as practically stationary at step number N^*, which in turn provides that $(M_s^{'})^{N^*}$, respectively $(M_s^{*})^{N^*}$, is sufficiently close to limit matrix.

Variance of the UD process

Regularity of matrices $M_s^{'}$ and M_s^{*} indicates that UD process stabilises exponentially i.e. it rapidly becomes stationary. We are looking for variance of this process when it is practically stationary.

Definition 1. We call UD process practically stationary after N steps, if $|DX_N(s)-DX_{N+l}(s)|<10^{-3}\sigma_0^2$ for all $l\geq 0$ and at the same step parity.

Thus the definition 1 requires that *N* and *l* both were even or both where odd numbers. Further in the analysis we use lemma, which is easy to prove.

Lemma 1. At every fixed UD process step, $s=\Delta\sigma_0$ and step number $n\geq 6$, an inequality holds as follows:

$$\frac{p_{\nu+2}(n,s)}{p_\nu(n,s)}<d_\nu\Phi(-(\nu+1)\Delta), \tag{11}$$

where $0\leq\nu\leq n-2$, $d_0=d_1=1.66$, $d_2=d_3=1.2$, $d_\nu=1$, if $\nu>3$.

Taking in account lemma 1 we can write:

$$DX_n=2\sum_{\nu=1}^{n}p_\nu(n,s)\nu^2s^2=$$

$$2\sum_{\nu=1}^{m}p_\nu(n,s)\nu^2s^2+2\sum_{\nu=m+1}^{n}p_\nu(n,s)\nu^2s^2, \tag{12}$$

where $6\leq m+1\leq n$.

Further in analysis we assume that *m+1* is even if n is even number, and *m+1* is odd if n is odd number. For convenience we denote the sum $\sum_{\nu=m+1}^{n}p_\nu(n,s)\nu^2$ as $\gamma(\Delta,m,n)$ and, using lemma1, we look for assessment from above for $\gamma(\Delta,m,n)$.

If $n\geq m+3$, then obviously

$$\gamma(\Delta,m,n)=p_{m+1}(n,s)(m+1)^2+p_{m+3}(n,s)(m+3)^2+ \\ +\ldots+p_n(n,s)n^2. \tag{13}$$

Since $m+1\geq 6$, $\Delta\geq 0.1$ and (see Lemma1)

$$\frac{p_{m+2k+3}(n,s)(m+2k+3)^2}{p_{m+2k+1}(n,s)(m+2k+1)^2}< \\ <(1+\frac{4}{m+2k+1}+\frac{4}{(m+2k+1)^2})\Phi(-(m+2k+2)\Delta), \tag{14}$$

where $k=0,1,\ldots,\frac{n-m-3}{2}$,

$$q(\Delta,m,k)=(1+\frac{4}{m+2k+1}+\frac{4}{(m+2k+1)^2})\Phi(-(m+2k+2)\Delta)<$$

$$<0.432. \qquad (15)$$

From (12) and (13) follows that

$$\gamma(\Delta,m,n)<(m+1)^2 p_{m+1}(n,s)\sum_{k=0}^{\infty}(q(\Delta,m,0))^k=$$

$$=(m+1)^2 p_{m+1}(n,s)\frac{1}{1-q(\Delta,m,0)}. \qquad (16)$$

The inequality (16) shows that calculating DX_n by formula $2\sum_{v=1}^{m} p_v(n,s)v^2s^2$ we produce error, the absolute value of which is less than $2s^2(m+1)^2 p_{m+1}(n,s)\frac{1}{1-q(\Delta,m,0)}$.

In result of calculations using formula (11) and (16) we obtain inequality:

$$2s^2\gamma(\Delta,13,n)<0.000272\sigma_0^2. \qquad (17)$$

By increasing Δ we can obtain the required error assessment at lesser m values. For instance, if $\Delta\geq 0.5$, $m=7$ will be sufficient for sure.

Further we present values of UD process variances for practically stationary case at various Δ values and step number parities. We denote the secondary variances as $\sigma_1^2(\Delta)$. In case of even step numbers the results are as follows: $\sigma_1^2(0.1)$ = $0.0641\sigma_0^2$, $\sigma_1^2(0.2)$ = $0.1333\sigma_0^2$, $\sigma_1^2(0.3)$ = $0.2061\sigma_0^2$, $\sigma_1^2(0.4)$ = $0.2810\sigma_0^2$, $\sigma_1^2(0.5)$ = $0.3613\sigma_0^2$, $\sigma_1^2(0.6)$ = $0.4378\sigma_0^2$, $\sigma_1^2(0.7)$ = $0.5081\sigma_0^2$, $\sigma_1^2(0.8)$ = $0.5677\sigma_0^2$, $\sigma_1^2(1)$ = $0.6468\sigma_0^2$.

Formula (17) shows that by all Δ values $(0.1\leq\Delta\leq 1)$ UD process becomes practically stationary after 16 steps.

Next we present practically stationary state probability distributions for UD process in case of even step numbers:

$P_0(16;0.1\sigma_0)$ = 0.3132, $P_2(16;0.1\sigma_0)$ = 0.2304, $P_4(16;0.1\sigma_0)$ = 0.0914, $P_6(16;0.1\sigma_0)$ = 0.0193, $P_8(16;0.1\sigma_0)$ = $2.1440\cdot10^{-3}$, $P_{10}(16;0.1\sigma_0)$ = $1.2147\cdot10^{-4}$, $P_{12}(16;0.1\sigma_0)$ = $3.4055\cdot10^{-6}$, $P_{14}(16;0.1\sigma_0)$ = $4,52\cdot10^{-8}$;

$P_0(16;0.2\sigma_0)$ = 0.4348, $P_2(16;0.2\sigma_0)$ = 0.2410, $P_4(16;0.2\sigma_0)$ = 0.0398, $P_6(16;0.2\sigma_0)$ = $1.7972\cdot10^{-3}$, $P_8(16;0.2\sigma_0)$ = $1.9221\cdot10^{-5}$, $P_{10}(16;0.2\sigma_0)$ = $4.0169\cdot10^{-8}$, $P_{12}(16;0.2\sigma_0)$ = $1.2991\cdot10^{-11}$, $P_{14}(16;0.2\sigma_0)$ = $4.9998\cdot10^{-14}$;

$P_0(16;0.3\sigma_0)$ = 0.5230, $P_2(16;0.3\sigma_0)$ = 0.2228, $P_4(16;0.3\sigma_0)$ = 0.0156, $P_6(16;0.3\sigma_0)$ = $1.3309\cdot10^{-4}$, $P_8(16;0.3\sigma_0)$ = $8.7703\cdot10^{-8}$, $P_{10}(16;0.3\sigma_0)$ = $2.5046\cdot10^{-10}$, $P_{12}(16;0.3\sigma_0)$ = $1.6355\cdot10^{-16}$, $P_{14}(16;0.3\sigma_0)$ = $1.2516\cdot10^{-24}$;

$P_0(16;0.4\sigma_0)$ = 0.5929, $P_2(16;0.4\sigma_0)$ = 0.1978, $P_4(16;0.4\sigma_0)$ = 0.0058, $P_6(16;0.4\sigma_0)$ = $7.414\cdot10^{-6}$, $P_8(16;0.4\sigma_0)$ = $1.558\cdot10^{-10}$, $P_{10}(16;0.4\sigma_0)$ = $1.7036\cdot10^{-15}$, $P_{12}(16;0.4\sigma_0)$ = $2.920\cdot10^{-25}$, $P_{14}(16;0.4\sigma_0)$ = $2.307\cdot10^{-38}$;

$P_0(16;0.5\sigma_0)$ = 0.6508, $P_2(16;0.5\sigma_0)$ = 0.1726, $P_4(16;0.5\sigma_0)$ = 0.0020, $P_6(16;0.5\sigma_0)$ = $2.855\cdot10^{-7}$, $P_8(16;0.5\sigma_0)$ = $8.968\cdot10^{-12}$, $P_{10}(16;0.5\sigma_0)$ = $9.650\cdot10^{-22}$, $P_{12}(16;0.5\sigma_0)$ = $5.255\cdot10^{-36}$.

To obtain these probabilities for odd step numbers the following formula may be used:

$$p_{2k+1}(17;\Delta\sigma_0)=p_{2k}(16;\Delta\sigma_0)\Phi(-2k\Delta)+$$

$$+p_{2k+2}(16;\Delta\sigma_0)\Phi((2k+2)\Delta). \qquad (18)$$

In case of odd step numbers the secondary variances $\breve{\sigma}_1^2(\Delta)$ may be calculated with the required accuracy using formula:

$$\breve{\sigma}_1^2(\Delta)=2s^2\sum_{k=0}^{\frac{n-1}{2}} p_{2k+1}(n,s)(2k+1)^2\approx$$

$$\approx 2s^2\sum_{k=0}^{8}(p_{2k}(16,s)\Phi(-2k\Delta)+$$

$$+p_{2k+2}(16,s)\Phi((2k+2)\Delta))(2k+1)^2 \qquad (19)$$

To make comparison of variances $\sigma_1^2(\Delta)$ and $\breve{\sigma}_1^2(\Delta)$ convenient, we transform formula (19) as follows:

$$\breve{\sigma}_1^2(\Delta)\approx 2s^2[p_2(16,s)(\Phi(0)\frac{p_0(16,s)}{p_2(16,s)}+\Phi(2\Delta)+$$

$$+9\Phi(-2\Delta)+p_4(16,s)(9\Phi(4\Delta)+25\Phi(-4\Delta))+$$

$$+p_6(16,s)(25\Phi(6\Delta)+49\Phi(-6\Delta))+\ldots+$$

$$+p_{16}(16,s)(225\Phi(16\Delta)+289\Phi(-16\Delta))]. \qquad (20)$$

where $\breve{\sigma}_1^2(0.1)$ $\approx 0.0646\sigma_0^2$, $\breve{\sigma}_1^2(0.2)$ $\approx 0.1333\sigma_0^2$, $\breve{\sigma}_1^2(0.5)$ $\approx 0.3679\sigma_0^2$, $\breve{\sigma}_1^2(0.6)$ $\approx 0.4630\sigma_0^2$, $\breve{\sigma}_1^2(1)\approx$ $\approx 1.02947\sigma_0^2$.

Comparison of these results with those published in literature [6, 7] shows that our results differ from the secondary variance estimates published in paper [7], which were found using formulas:

$$\begin{cases}\sigma_1^2(\Delta)=0.625\mathrm{s}\,\sigma_0+0.25\mathrm{s}^2, \\ \mathrm{S}=\Delta\sigma_0.\end{cases} \qquad (21)$$

To solve efficiently problems of digital signal processing, knowledge of UD process variance alone is not enough. There are also other statistical characteristics of importance. Further research has to deal with issues of UD process covariance specifics. Preliminary analysis suggests that in practically stationary situation at step length $s=0,5\sigma_0$ function $e^{-0.42t-1}\sigma_0^2$ is good approximation for the covariance, where t is shift between step numbers, i.e. $\mathrm{cov}(X_N,X_{N+t})\approx e^{-0.42t-1}\sigma_0^2$.

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Mathematical model for UD (Up and Down) procedure of digital signal processing was formulated as a simple homogeneous Markov chain. For such Markov chains were investigated transition probability distributions by step numbers sufficient for practically stationary process. Conditions were analysed under which infinite Markov chain may be replaced by a finite Markov chain practically equivalent to the infinite chain, i.e. difference between variances of both processes is negligible. It was shown how these variances depend on UD procedure step length and step number parity. Bibl. 7 (in English; summaries in English, Russian and Lithuanian).

А. Лоренцс. UD метод цифровой обработки сигналов и статистические его особенности // Электроника и электротехника. – Каунас: Технология, 2008. – № 6(86). – С. 33–36.

Математическая модель процедуры UD (англ. Up and Down) обработки цифрового сигнала была сформулирована как простая гомогенная цепь Маркова. Для такой цепи были исследованы распределения вероятности шагового перехода, когда количество шагов достаточно для стационарного процесса. Были проанализированы условия, под которыми бесконечная цепь Маркова может быть заменена конечной цепью Маркова, фактически эквивалентной бесконечной цепи, то есть разницы между вариациями обоих процессов незначительны. Показано, как эти разницы зависят от длины шага процедуры UD и четности номера шага. Библ. 7 (на английском языке; рефераты на английском, русском и литовском яз.).

A. Lorencs. Skaitmeninių signalų apdorojimo UD metodas ir jo statistinės charakteristikos // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2008. – Nr. 6(86). – P. 33–36.

Suformuluotas skaitmeninių signalų apdorojimo UD (angl. Up and Down) procedūros matematinis modelis ir paprasta homogeninė Markovo grandis. Tirti pereinamieji tikimybių skirstiniai, kurių pakopų skaičius pakankamas praktikoje pasitaikantiems stacionariesiems procesams. Analizuotos sąlygos, kuriomis begalinė Markovo grandis gali būti pakeista ekvivalenčia baigtine Markovo grandimi, t. y. skirtumas tarp abiejų procesų nuokrypių yra nykstamai mažas. Parodyta, kaip šie nuokrypiai priklauso nuo UD procedūros pakopos ilgio ir žingsnio numerio lyginumo. Bibl. 7 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).