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Algorithmic Methods of Variational Calculus

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Introduction

Problems of automatic control system (ACS) synthesis and optimization are solved using variational calculus, maximum principle, dynamic programming and other classical methods of ACS synthesis. Problems of optimal control are often solved using variational calculus methods which are simple and convenient according to the authors of the monograph [1]. However the indicated methods are not universal; it is difficult to apply them when the object is described by logic operators and impossible when mathematical model does not exist.

The objective of the present study is as follows: by application of search optimization methods [2] and system synthesis methods [3] to create algorithmic variational calculus methods that would allow to solve variational calculus problems in cases when mathematical model (functional) of the object is not set by analytic method, and it is impossible to apply classical synthesis methods (including variational calculus). This work is a sequel to the article [4].

Problems of variational calculus with unfixed trajectory ends

For problems with unfixed marginal trajectory ends (x_0, y_0) and (x_1, y_1) they can change their location and can be located in any points of given lines or surfaces

$$y(x_0) \in \varphi(x), \quad y(x_1) \in c(x). \tag{1}$$

An extremal needs to be found that would give minimum to the functional

$$J = \int_{x_0}^{x_1} f(x, y, \dot{y}) dx. \tag{2}$$

Euler's equation is applied for solving the problem (1), (2)

$$f_y - \frac{d}{dx} f_{\dot{y}} = 0 \tag{3}$$

with conditions

$$\left[f + (\dot{\varphi} - \dot{y}) f_{\dot{y}} \right]_{x=x_0} = 0, \tag{4}$$

$$\left[f + (\dot{c} - \dot{y}) f_{\dot{y}} \right]_{x=x_1} = 0, \tag{5}$$

where $f_y = \frac{df}{dy}$, $f_{\dot{y}} = \frac{df}{d\dot{y}}$.

Integration constants and particular solution can be found from the equations (4) and (5), which are known as transversal conditions.

Tasks with the unfixed primary or final end of the trajectory are also possible. In such case, the tasks are solved by application of one of the (4) and (5) conditions. It must be observed, that solution of the task by application of Eiler‘s equation (3) on the conditions (4), (5) requires analytical expression of the function $f(x, y, \dot{y})$. Moreover, this function must have partial fluxions, which are continuous up to the second row on the basis of all variables.

Extremals with corner points

It is estimated [1], that extremal of the functional (2) is a smooth curve if $f(x, y, \dot{y})$ is a continuous function of its arguments which has second row fluxion that is not equal to zero, i.e. $f_{\dot{y}\dot{y}} \neq 0$. The functional can reach extremum when extremal has corners if in separate points $f_{\dot{y}\dot{y}} = 0$.

It is obvious that function $y(x)$ which provides extremum to the functional must meet Euler's equation between the corner points. Let function $y(x)$ be an extremal with one corner in point x_1 which is located between the points x_0 and x_2. The functional

$$J = \int_{x_0}^{x_2} f(x, y, \dot{y}) dx; \; y(x_0) = y_0; \; y(x_2) = y_2 \tag{6}$$

can be expressed as a sum of integrals

$$J = J_1 + J_2 = \int_{x_0}^{x_1} f(x, y, \dot{y}) dx + \int_{x_1}^{x_2} f(x, y, \dot{y}) dx. \tag{7}$$

The variation of the functional (7) can be expressed as the sum of variations δJ_1 and δJ_2.

Function $y(x)$ is extremal in each interval $[x_0, x_1]$ and $[x_1, x_2]$ and meets the requirements of Euler's equation (3). Consequently

$$\delta J_1 + \delta J_1 = 0\,. \qquad (8)$$

The following conditions were obtained from (8):

$$\begin{cases} f_{\dot{y}}\big|_{x=x_1-0} = f_{\dot{y}}\big|_{x=x_1+0}\,, \\ \left[f - \dot{y}f_{\dot{y}} \right]_{x=x_1-0} = \left[f - \dot{y}f_{\dot{y}} \right]_{x=x_1+0}\,. \end{cases} \qquad (9)$$

Conditions (9) (Veierstrass-Erdman conditions) together with Euler's equation enable finding the extremal of the functional (6) in such cases when earlier stated requirements for the function $f(x, y, \dot{y})$ are met.

Algorithmic Methods of Variational Calculus

We shall analyze the solution of problem (1), (2) by application of methods of algorithmic system synthesis [3, 4].

Within the interval $x_0 \le x \le x_1$ when $x_0 = 0$, using discreet values of function $y(x)$

$$y[iT]\,,\ i = 0, ..., N-1\,, \qquad (10)$$

a k – dimensional vector z is introduced

$$\begin{cases} z = \{ z_1 = y[0],\ z_2 = y[1T], ..., \\ \quad z_{k-1} = y[(N-2)T],\ z_k = y[(N-1)T] \}, \end{cases} \qquad (11)$$

where $k = N$, $T = t_f / N$ is a sampling period.

A step function or another function made out of linear intervals is formed using the components of vector z

$$y = y(z, x)\,,\ x_0 \le x \le x_1\,. \qquad (12)$$

Then a variational calculus problem (1), (2) becomes a search optimization problem. An extremal $y(x)$ has to be found that would secure functional

$$J(z) = J[y(z, x)]\,,\ x_0 \le x \le x_1 \qquad (13)$$

minimum with respect to marginal conditions

$$y(x_0) \in \varphi(x)\,,\quad y(x_1) \in c(x)\,. \qquad (14)$$

There is a set of extremals meeting various marginal conditions and the task (13), (14) can be solved e.g. m times. In this way m local extremums $Z_1^*, ..., Z_m^*$ are found during the search process. A vector among local extremums is selected which gives minimal value the functional $J(z)$

$$z_{\min}^* = \arg\min I(z_j^*)\,,\ j = 1, ..., m\,. \qquad (15)$$

It is obvious that when $m \to \infty$ a probability that $Z_{\min}^*$ is a global minimum Z^{**} is tending toward one:

$$\lim_{x \to \infty} P\left\{ \left\| z_{\min}^* - z^{**} \right\| \le v_0 \right\} = 1\,, \qquad (16)$$

where v_0 - given error of optimization.

Solution process of task (13), (14) can be automated using two optimizers. One to optimize the shape of curve $y(x)$ another to search for optimal marginal conditions (m. c).

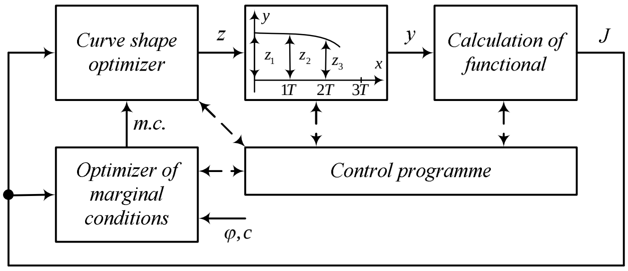

Fig. 1. Scheme of solution for problem with unfixed trajectory ends

Variational calculus problem (13), (14) is solved applying the methods of simplex search [2] according to the scheme in Fig. 1.

We shall analyze the solution of task (6) applying algorithmic methods of system synthesis [3, 4].

We can expect to have a problem with extremal which has corner points if preliminary investigation shows that in separate points $f_{\dot{y}\dot{y}} = 0$.

We will formulate a task for search optimization.

Within the interval $x_0 \le x \le x_1$ when $x_0 = 0$, using discreet values of function $y(x)$ (10) a k – dimensional vector (11) is introduced. A step function is formed using the components of vector z

$$y = y(z, x)\,,\ 0 \le x \le x_2\,. \qquad (17)$$

Problem of variational calculus (6) is written in the form of search optimization. An extremal $y(x)$ has to be found that would secure functional

$$J(z) = J[y(z, x)]\,,\ 0 \le x \le x_2 \qquad (18)$$

minimum with respect to marginal conditions

$$y(x_0) = y_0\,;\ y(x_2) = y_2\,. \qquad (19)$$

Variational calculus problem (18), (19) is solved applying the methods of simplex search [2] according to the scheme in Fig. 2.

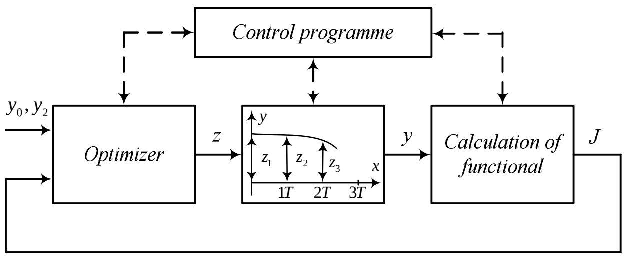

Fig. 2. Scheme for search of extremal with corner points

Examples of variational calculus problem solutions following algorithmic methods

Problem No. 1. A curve connecting two points A and B must be found such that sliding point mass will reach the end of curve at shortest time (friction is ignored). Starting point A is fixed, end point B can move vertically. This is a classical problem of variational calculus the solution of which is a curve called brachistochrone (from Greek *brachistos* „the shortest“ and *chronos* “time”).

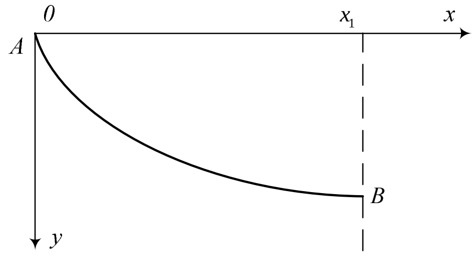

Fig 3. A curve of problem No. 1

According to [1] this tas kis formulated in the following way: a function $y(x)$ needs to be found that would secure minimum to the functional

$$J(y) = \int_0^{x_1} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} dx . \qquad (20)$$

Marginal conditions are given by:

$$y(0) = 0 , \; x_1 = 1 . \qquad (21)$$

Problem (20), (21) is solved following the technique (10)–(16) and the scheme of Fig. 1. A one-dimensional simplex is employed in the line $x_1 = 1$ for the search of optimal marginal conditions. For the search of curve shape we choose that N=51 then

$$T = \frac{t_f}{N} = 0,02 .$$

A 51-gonal simplex is employed in 50-dimensional space of variables for the search of curve shape. Optimal marginal conditions $y_1(x_1)$ and corresponding curve $y(t)$ is searched using simplex search method of forbidden backward step. The result of problem solution is extremal $y(x)$ found by algorithmic method. Fig. 4 shows the extremal $y(x)$ found by algorithmic method (curve No. 1), theoretical extremal (curve No. 2) and several extremals found during search process when not optimal marginal conditions are set (curves No. 3-6), the values of functional are also presented for different marginal conditions.

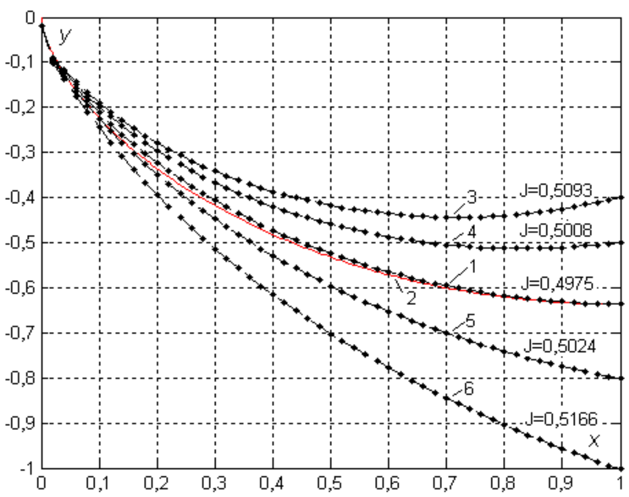

Fig 4. A chart of extremals of problem No. 1

Problem No. 2. A curve needs to be found that would give minimum to the functional

$$J = \int_0^2 y^2 (1-\dot{y})^2 dx . \qquad (22)$$

Marginal conditions are given by:

$$y(0) = 0 , \; y(2) = 1 . \qquad (23)$$

Problem (22), (23) is solved following the technique (17)-(19) and the scheme of Fig. 2. We choose that $N = 21$ then

$$T = \frac{t_f}{N} = 0,1 . \qquad (24)$$

A 21-gonal simplex is employed in 20-dimensional space of variables for the task solution. The extremal $y(x)$ is searched using simplex search method of forbidden backward step. The result of problem solution is extremal $y(x)$ found by algorithmic method. Fig. 5 shows the extremal $y(x)$ found by algorithmic method (curve No. 1) and theoretical extremal (curve No. 2). An extremal has a corner in point $y(1) = 0$.

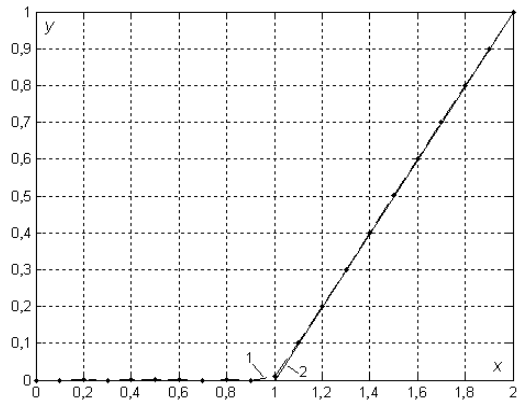

Fig. 5. A chart of extremals of problem No. 2

Problem No. 3. A curve needs to be found that would give minimum to the functional

$$J = \int_0^2 (1-y)^2 (1+\dot{y})^2 dx . \qquad (25)$$

Marginal conditions are given by:

$$y(0) = 1 , \; y(2) = 0 . \qquad (26)$$

Problem (25), (26) is solved following the technique (17)-(19) and the scheme of Fig. 2. We choose that $N = 21$ then

$$T = \frac{t_f}{N} = 0,1 . \qquad (27)$$

A 21-gonal simplex is employed in 20-dimensional space of variables for the task solution. The extremal $y(x)$ is searched using simplex search method of forbidden backward step. The result of problem solution is extremal $y(x)$ found by algorithmic method. Fig. 6 shows the extremal $y(x)$ found by algorithmic method (curve No. 1) and theoretical extremal (curve No. 2). An extremal has a corner in point $y(1) = 1$.

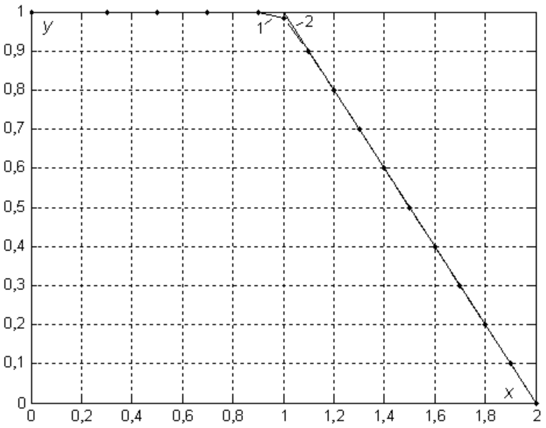

Fig. 6. A chart of extremals of problem No. 3

Conclusions

Algorithmic methods of variational calculus are created that allow solving various variational calculus problems with unfixed trajectory ends by applying simplex search algorithms. Extremals can be found during search optimization (including extremals with corner points) even in such cases when mathematical model of the object (functional) is described by logic operators, or its analytical expression is unknown, i.e. in cases when classical variational calculus methods are impossible to apply.

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In problems of automatic control system optimization it is required to determine the structure of controller, parameters or the law of reference value variation which would guarantee the required quality of control. Methods of variational calculus are often used to solve problems of optimal control when control objects are simple and have mathematical models. But these methods are not universal, it is difficult to use them when objects are defined by logical operators and it is impossible to use them when mathematical model does not exist. The aim of the present work is as follows: by application of optimization methods to create algorithmic variational calculus methods that would allow solving variational calculus problems in cases when mathematical model (functional) of the object is not set by analytical method and it is impossible to apply classical methods. The technique of algorithmic variational calculus method is set in the article, problems of variational calculus with unfixed trajectory ends are formulated in the form of search optimization problems, methods of finding extremals with corners are indicated and examples of solutions of variational calculus problems are presented. Ill. 6, bibl. 4 (in Lithuanian; summaries in English, Russian and Lithuanian).

А. Дамбраускас, В. Ринкевичюс. Алгоритмические методы вариационного исчисления // Электроника и электротехника. – Каунас: Технология, 2008. – № 5(85) – С. 25–28.

При решении задач оптимизации автоматических систем управления необходимо установить структуру и параметры устройства управления, или закон управляющего воздействия, которые обеспечило бы необходимое качество управления. Когда объекты управления простые и имеют математические модели, часто для решения задач оптимального управления применяют методы вариационного исчисления. И всё же указанные методы не являются универсальными, их применение затруднительно, когда объект описан логическими операторами и невозможно, когда математической модели вообще нет. Цель этой работы, используя методы поисковой оптимизации, создать алгоритмические методы вариационного исчисления, позволяющие решать задачи вариационного исчисления в тех случиях, когда математическая модель объекта (функционал) аналитической форме не задана, когда применение классических методов вариационного исчисления невозможна. В статье изложена алгоритмическая методика вариационного исчисления, сформулированы задачи вариационного исчисления с подвижными концами траектории в форме задач поисковой оптимизации, указаны способы поиска экстремаль с изломами, приведены примеры решения задач вариационного исчисления. Ил. 6, библ. 4 (на английском языке; рефераты на английском, русском и литовском яз.).

A. Dambrauskas, V. Rinkevičius. Algoritminiai variacinio skaičiavimo metodai // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2008. – Nr. 5(85). – P. 25–28.

Sprendžiant automatinių valdymo sistemų optimizavimo uždavinius reikia nustatyti valdymo įtaiso struktūrą, parametrus arba valdymo poveikio kitimo dėsnį, kurie užtikrintų reikiamą valdymo kokybę. Kai valdymo objektai yra paprasti ir turi matematinius modelius, optimalaus valdymo uždaviniams spręsti dažnai naudojami variacinio skaičiavimo metodai. Tačiau šie metodai nėra universalūs, juos taikyti keblu, kai objektas aprašytas loginiais operatoriais, ir neįmanoma, kai matematinio modelio iš viso nėra. Šio darbo tikslas, taikant optimizavimo metodus, kurti algoritminius variacinio skaičiavimo metodus, leidžiančius spręsti variacinio skaičiavimo uždavinius tais atvejais, kai objekto matematinis modelis (funkcionalas) analitiniu būdu nenurodytas, kai klasikinių skaičiavimo metodų taikyti neįmanoma. Straipsnyje išdėstyta algoritminė variacinio skaičiavimo metodika, suformuluoti variacinio skaičiavimo uždaviniai su nefiksuotais trajektorijos galais paieškinio optimizavimo uždavinių forma, nurodyti ekstremalių su lūžiais radimo būdai, pateikta variacinio skaičiavimo uždavinių sprendimo pavyzdžių. Il. 6, bibl. 4 (lietuvių kalba; santraukos anglų, rusų ir lietuvių k.).

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