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# Algorithmic Methods of Variational Calculus

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## Introduction

In problems of automatic control system (ACS) optimization it is required to determine the structure of controller, its parameters or the law of reference value variation which would guarantee the required quality of control. Problems of optimal control can be solved using variational calculus, maximum principle, dynamic programming and other classical methods of ACS synthesis when control objects are simple and have mathematical models. Problems of optimal control are often solved using variational calculus methods which are simple and convenient according to the authors of the monograph (1). However the indicated methods are not universal; it is difficult to apply them when the object is described by logic operators and impossible when mathematical model does not exist.

The objective of the present study is as follows: by application of search optimization methods [2] and system synthesis methods [3] to create algorithmic variational calculus methods that would allow to solve variational calculus problems in cases when mathematical model (functional) of the object is not set by analytic method, and it is impossible to apply classical synthesis methods (including variational calculus).

## Problems of variational calculus

We are going to analyze several simple problems of variational calculus.

The simplest classical problem of variational calculus is formulated as follows: out of a set of functions an extremal $y(t)$ needs to be found that would give the functional

$$I = \int_{t_0}^{t_f} f_0(t, y, \dot{y}) dt \qquad (1)$$

minimum and would intersect fixed marginal trajectory ends

$$y(t_0) = y_0 ,\ y(t_f) = y_f , \qquad (2)$$

where $f_0(t, y, \dot{y})$ – function of variables $t, y$ and $\dot{y}$ that is generally uninterrupted and has uninterrupted partial fluxions up to the second row following all variables.

Euler's equation is applied for solving the problem (1), (2)

$$\frac{\partial f_0(t, y, \dot{y})}{\partial y} - \frac{d}{dt}\frac{\partial f_0(t, y, \dot{y})}{\partial \dot{y}} = 0 .$$

Its solution is the searched extremal $y(t)$.

Problems of variational calculus that require finding extremals for their solutions make a certain class of problems. Problems with fixed trajectory ends, problems with unfixed trajectory ends, search of extremals with breaking points can be attributed to it.

Another class of variational problems involves problems connected with finding conditional extremum of a functional, e.g. function $y(t)$ needs to be found that would give the functional

$$I = \int_{t_0}^{t_f} f_0(t, y, \dot{y}) dt \qquad (3)$$

minimum with respect to limitations

$$h_j(t, y) = 0 ,\ j = 1 ,\dots, p < n , \qquad (4)$$

where $y$ –is an $n$-dimensional vector.

The method of Lagrange multipliers is applied for solution (3), (4). Lagrange function is written down

$$F(t, y, \dot{y}) = f_0(t, y, \dot{y}) + \sum_{j=1}^{p} \lambda_j h_j(t, y) . \qquad (5)$$

A system of $n+p$ equations is made, where $n$ of Euler's equations

$$\frac{\partial F(t, y, \dot{y})}{\partial y_i} - \frac{d}{dt}\frac{\partial F(t, y, \dot{y})}{\partial \dot{y}_i} = 0 ,\ i = 1 ,\dots,n \qquad (6)$$

and $p$ limitation equations are involved (4).

From the equation system (4), (6) a solution $y_1(t) ,\dots, y_n(t)$ is found which corresponds to conditional extremum of the functional (3).

It has to be noted that application of methods of Euler's equations and Lagrange multipliers for the solution of problems (1), (2) and (3), (4) requires analytical expression of function $f_0(t, y, \dot{y})$; in addition this function must have uninterrupted partial fluxions.

### The basics of algorithmic methods of variational calculus

We shall analyze the solution of problem (1), (2) by application of methods of algorithmic system synthesis [3].

Within the time interval $t_0 \le t \le t_f$ when $t_0$=0, using discreet values of function $y(t)$

$$y[iT]\,, i = 0\,,...,\, N-1\,, \qquad (7)$$

a $k$ – dimensional vector $x$ is introduced.

$$x = \{x_1 = y[0], x_2 = y[1T],..., \\ x_{k-1} = y[(N-2)T], x_k = y[(N-1)T]\}, \qquad (8)$$

where $k = N$, $T = t_f / N$ is a sampling period.

A step function or another function made out of linear intervals is formed using the components of vector $x$

$$y = y(x,t),\ 0 \le t \le t_f\,. \qquad (9)$$

Then a variational calculus problem (1), (2) becomes a search optimization problem. An extremal $y(t)$ has to be found that would secure functional

$$I(x) = I[(y(x,t)],\ 0 \le t \le t_f \qquad (10)$$

minimum with respect to marginal conditions

$$y(t_0) = y(0) = y_0\ ;\ y(t_f) = y_f\,. \qquad (11)$$

Problem (10), (11) is solved applying the methods of simplex search [2]. The scheme of problem solution is presented in Fig. 1.

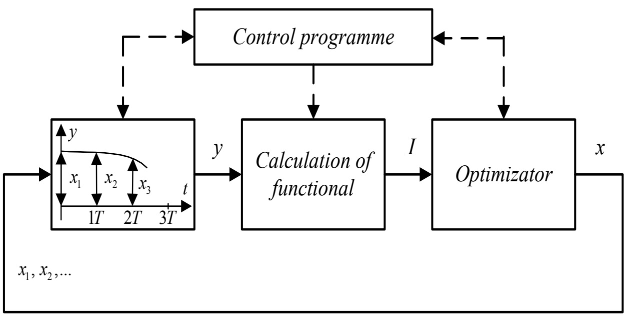


**Fig. 1.** Scheme of variational calculus problem solution

Variational calculus problem (3), (4) i.e. optimization problem with functional limitations in the form of equals (4), can be rearranged into an optimization problem without limitations with the help of penalty function method. This problem can be formulated as follows:

$$I_1(x) = I(x) + P(x) = I[y(x,t)] + \\ + \sum_{j-1}^{p} \varphi_j h_j^2 [y(x,t)] \to \min\,, \qquad (12)$$

where $P(x)$- penalty function; $\varphi_j$ – weight coefficient, $y(x,t)$ – vector function.

Problem (12) is solved by application of simplex search methods following scheme of Fig. 2.

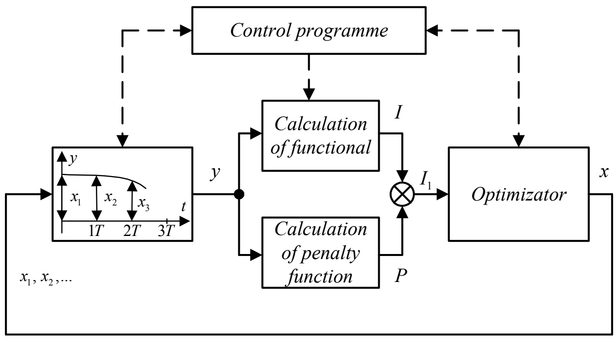


**Fig. 2.** Scheme of solution of variational calculus problem with functional limitations

### Examples of variational calculus problem solutions following algorithmic methods

*Problem No. 1*. Function $y(t)$ needs to be found that would give a minimum to the functional

$$I = \int_0^{\frac{\pi}{2}} f_0(t, y, \dot{y})dt = \int_0^{\frac{\pi}{2}} (y^2 - \dot{y}^2)dt\,. \qquad (13)$$

Marginal conditions are given by

$$y(0) = 1\,,\ y(\pi/2) = 0\,. \qquad (14)$$

Problem (13), (14) is solved following the technique (7)-(11) and the scheme of Fig. 1. We choose that $N$=8 then

$$T = \frac{t_f}{N} = 0{,}196s\,.$$

Since the marginal points $y(0)$ and $y(NT)$ of the trajectory are fixed, an octangular simplex is made in seven dimensional space. Extremal $y(t)$ is searched according to simplex search method of prohibited backward step. The result of problem solution – extremal $y(t)$ found by algorithmic means. It is shown in Fig. 3 (curve No. 1). Theoretical extremal $y(t)$ [3] is shown in Fig. 3 (curve No. 2).

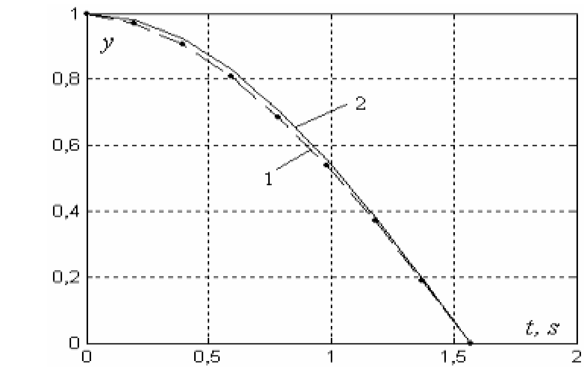


**Fig. 3.** A chart of extremals of problem No. 1

*Problem No. 2*. Function $y(t)$ needs to be found that would give a minimum to the functional (13) complying with marginal conditions

$$y(0) = 0\,,\ y(\pi/2) = 1\,. \qquad (15)$$

The problem is solved the same way as indicated above. The result of problem solution – extremal $y(t)$ found by algorithmic method. It is shown in Fig. 4 (curve No. 1). Theoretical extremal $y(t)$ [3] is shown in Fig. 4 (curve No 2).

*Problem No. 3.* Function $y(t)$ needs to be found that would give a minimum to the functional

$$I = \int_0^1 \left(1 + \ddot{y}^2\right) dt . \qquad (16)$$

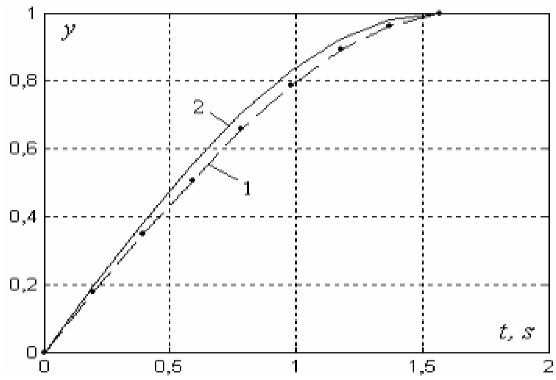


**Fig. 4.** A chart of extremals of problem No. 2

Marginal conditions are given

$$y(0) = 0\,,\ \dot{y}(0) = 1\,,\ y(1) = 1\,,\ \dot{y}(1) = 1\,. \qquad (17)$$

Problem (16) is solved (17) following the same technique (7)-(11) and scheme of Fig. 1. We choose that $N$=5 then

$$T = \frac{t_f}{N} = 0{,}2s\,.$$

A pentagonal simplex is made in four dimensional space of variables. Extremal $y(t)$ is searched according to simplex search method of backward step. The result of problem solution – extremal $y(t)$ found by algorithmic means is shown in Fig. 5 (curve No. 1). Theoretical extremal $y(t)$ [3] is shown in Fig. 5 (Curve No. 2).

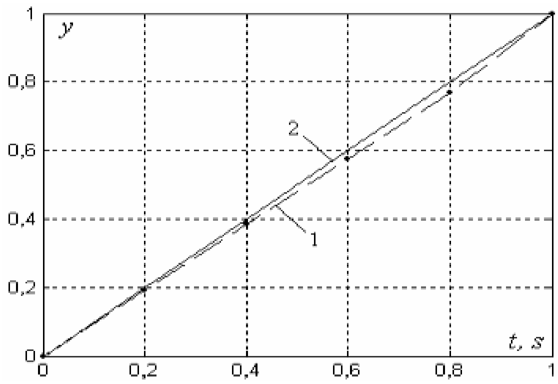


**Fig. 5.** A chart of extremals of problem No. 3

*Problem No. 4.* The equation of torque of DC motor

$$I\frac{d\omega}{dt} = M_d - M_c\,, \qquad (18)$$

where $M_d = C_M i$ – dynamic torque (motor torque); $M_c$ – load torque; $I$- inertia; $C_M$ – coefficient.

From (18) we get

$$i = \frac{M_d}{C_M} = \frac{1}{C_M}\left(I\frac{d\omega}{dt} + M_c\right). \qquad (19)$$

Heating of motor during time interval $T$ when $M_c$ =0

$$Q = \int_0^T Ri^2 dt = \frac{I^2 R}{C_M^2}\cdot\int_0^T \dot{\omega} dt = k\int_0^T \dot{\omega}^2 dt\,, \qquad (20)$$

where $R$ is resistance of the armature; $k = \dfrac{I^2 R}{C_M{}^2}$.

The problem is formulated as follows: laws of speed $\omega(t)$ and current $i(t)$ variation have to be found that would assure maximum angular displacement of the shaft

$$\varphi(\omega) = \int_0^T \omega dt = \int_0^T f_0(t,\omega,\dot{\omega})dt \to \max\,, \qquad (21)$$

keeping the limitations

$$Q(\omega) = k\int_0^T \dot{\omega}^2 dt = k\int_0^T f_1(t,\omega,\dot{\omega})dt = Q_0\,. \qquad (22)$$

Problem (21), (22) is solved applying the method of Lagrange multipliers.

However when analytical expressions of functions $f_0$ and $f_1$ are unknown or are very complicated, problem (21), (22) can be solved according to algorithmic variational calculus methods if there is a possibility to measure or calculate funcionals $\varphi(\omega)$ and $Q(\omega)$. Problem (21), (22) will be rearranged using the penalty function method: law $\omega(t)$ needs to be found which gives maximum to the functional $I_1(\omega)$

$$I_1(\omega) = \varphi[\omega(x,t)] - \left\{Q[\omega(x,t)] - Q_0\right\}^2 a \to \max\,, \quad (23)$$

following marginal conditions $\omega(0)$=0, $\omega(T)$=0 (here $a$ – weight coefficient; $\omega(x,t)$ – step function). Law $i(t)$ can be found from equation $i = I\dot{\omega}/C_M$, or inserting to (23) $i$ instead of $\omega$.

Problem (23) is solved following an identical scheme as the one shown in Fig. 2. The solution of problem (23) is shown in Fig. 6 – it is a speed law $\omega(t)$ found by algorithmic method (curve No. 1) and theoretical optimal law $\omega(t)$ [4] (curve No. 2). The chart of armature current of motor working in the mode of curve No. 1 from Fig. 6 is shown in Fig. 7 (curve No. 1).

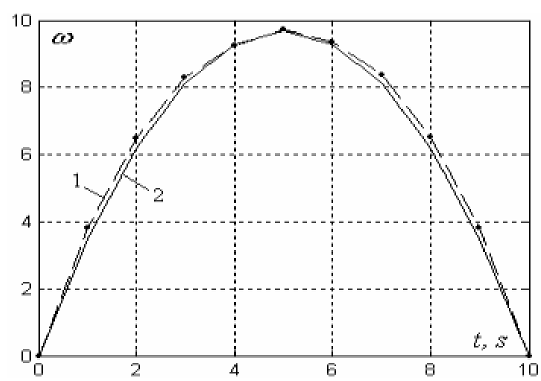


**Fig. 6.** Charts of speed extremals of problem No. 4

Theoretical law $i(t)$ is shown in Fig. 7 (curve No. 2).

Analytical expressions of funcionals were not used in the process of solving problems (Fig. 3–6).

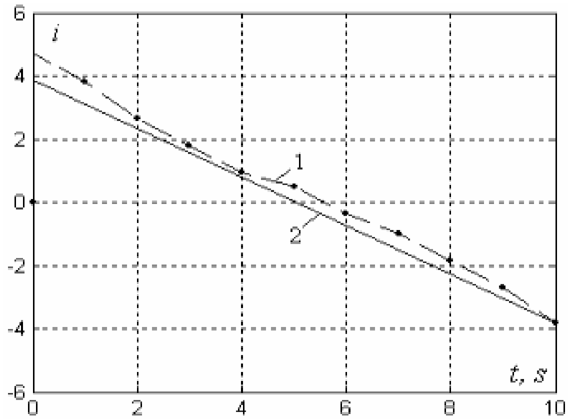


**Fig. 7.** Charts of armature current of problem No. 4

## Conclusions

The created algorithmic methods of variational calculus allow to solve various variational calculus problems by applying simplex search algorithms, e.g. to find extremals during search optimization, even in such cases when mathematical model of the object (functional) is described by logic operators, or its analytical expression is unknown, i.e. in cases when classical variational calculus methods are impossible to apply.

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**A. Dambrauskas, V. Rinkevičius. Algorithmic Methods of Variational Calculus // Electronics and Electrical Engineering. – Kaunas: Technologija, 2007. – No. 6(78). – P. 75–78.**

In problems of automatic control system optimization it is required to determine the structure of controller, parameters or the law of reference value variation which would guarantee the required quality of control. Methods of variational calculus are often used to solve problems of optimal control when control objects are simple and have mathematical models. But these methods are not universal, it is difficult to use them when objects are defined by logical operators and it is impossible to use them when mathematical model does not exist. The aim of the present work is as follows: by application of optimization methods to create algorithmic variational calculus methods that would allow solving variational calculus problems in cases when mathematical model (functional) of the object is not set by analytical method and it is impossible to apply classical methods. The technique of algorithmic variational calculus method is set in the article, problems of variational calculus are formulated in the form of search optimization problems, methods of solution are indicated and examples of solutions of variational calculus problems are presented. Ill. 7, bibl. 4 (in Lithuanian; summaries in English, Russian and Lithuanian).



**А. Дамбраускас, В. Ринкевичюс. Алгоритмические методы вариационного исчисления // Электроника и електротехника. – Каунас: Технология, 2007. – № 6(78). – С. 75–78.**

При решении задач оптимизации автоматических систем управления необходимо установить структуру и параметры устройства управления или закон управляющего воздействия, которые обеспечили бы необходимое качество управления. Когда объекты управления простые и имеют математические модели, часто для решения задач оптимального управления применяют методы вариационного исчисления. И всё же указанные методы не являются универсальными, их применение затруднительно, когда объект описан логическими операторами и невозможно, когда математической модели вообще нет. Цель этой работы – используя методы поисковой оптимизации, создать алгоритмические методы вариационного исчисления, позволяющие решать задачи вариационного исчисления в тех случиях, когда математическая модель объекта (функционал) в аналитической форме не задан, когда применение классических методов вариационного исчисления невозможно. В статье изложена алгоритмическая методика вариационного исчисления, сформулированы задачи вариационного исчисления в форме задач поисковой оптимизации, указаны способы решения, приведены примеры решения задач вариационного исчисления. Ил. 7, библ. 4 (на литовском языке; рефераты на английском, русском и литовском яз.).



**A. Dambrauskas, V. Rinkevičius. Algoritminiai variacinio skaičiavimo metodai // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2007. – Nr. 6(78). – P. 75–78.**

Sprendžiant automatinių valdymo sistemų optimizavimo uždavinius, reikia nustatyti valdymo įtaiso struktūrą, parametrus arba valdymo poveikio kitimo dėsnį, kurie užtikrintų reikiamą valdymo kokybę. Kai valdymo objektai yra paprasti ir turi matematinius modelius, dažnai optimalaus valdymo uždaviniams spręsti naudojami variacinio skaičiavimo metodai. Tačiau šie metodai nėra universalūs, juos taikyti keblu kai objektas aprašytas loginiais operatoriais, ir neįmanoma, kai matematinio modelio išvis nėra. Šio darbo tikslas – taikant optimizavimo metodus, kurti algoritminius variacinio skaičiavimo metodus, leidžiančius spręsti variacinio skaičiavimo uždavinius tais atvejais, kai objekto matematinis modelis (funkcionalas) analitiniu būdu neduotas, kai klasikinių skaičiavimo metodų taikyti neįmanoma. Straipsnyje išdėstyta algoritminė variacinio skaičiavimo metodika, suformuluoti variacinio skaičiavimo uždaviniai paieškinio optimizavimo uždavinių forma, nurodyti sprendimo būdai, pateikta variacinio skaičiavimo uždavinių sprendimo pavyzdžių. Il. 7, bibl. 4 (lietuvių kalba; santraukos anglų, rusų ir lietuvių k.).

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