ELECTRONCS AND ELECTRICAL ENGINEERING
ISSN 1392 – 1215 2006. Nr. 3(67)
ELEKTRONIKA IR ELEKTROTECHNIKA

T 180 TELECOMMUNICATION ENGINEERING
TELEKOMUNIKACIJŲ INŽINERIJA

Performance Measures Analysis of *M*/*M*/*m*/*K*/*N* Systems with Finite Customer Population

R. Rindzevičius, D. Poškaitis, B. Dekeris
Department of Telecommunications, Kaunas University of Technology,
Studentu str. 50-450, LT-51306 Kaunas, Lithuania; tel. (+370 37) 300515; e–mail: ramutis.rindzevicius@tef.ktu.lt

Introduction

There are some classical teletraffic methods used for analysis of telecommunication systems with finite customer population. Markovian models with interarrival and service times both are exponentially distributed. They have been studied for a long time with the development of telephone exchanges, and now they have been widely applied in data networks performance measures evaluations. In section 1 this paper presents a traditional wide performance measures analysis of system with finite buffer capacity and finite number of call sources. In section 2 it presents the analysis of an asymmetric system performance measures. The solution to our *M*/*M*/*m*/*K*/*N* system, which is a Markov birth-death system, may be obtained by applying the general solution given by L.Kleinrock [1]. A Markov queuing system is one which is characterized by a Poisson arrival process and exponentially distributed service times. Queuing models for *M*/*M*/*m*/*K*/*N* systems are very elegant in analysis and provide good closed results. Some of these models were extensively used in designing telephone networks, but they are also very useful in modeling of packets or messages services in data networks, as the packets or messages length usually lies between constant and exponential distribution. The *M*/*M*/*m*/*K*/*N* queue system and its variant models can be used to model the arrival of messages or packets in data networks at transmission channel with finite input buffer. In this case the service time is the packet length in bits divided by the channel transmission rate. The offered Engset queuing loss system models can be viewed as a generic model for variety of switches or exchanges in a circuit-switched network with finite arriving calls. Some authors in [1,2,3,4] provide analysis of the systems with finite customer population, various servicing protocols and system structures.

In symmetric queuing system the response time depends only on whether an arrival packet finds a free channel, but in asymmetric system it differs because the channels transmission bit rates are not equal and depend on which of the channels transmits the packet.

Finite customer population queuing system *M/M/m/K/N* performance measures

Let us consider a finite customer population system with m transmission channels with full availability and finite buffer of size K**,** exponential servicing time with mean value $1/\mu$. Assume α to be the arrival data packet rate from an idle call source. Then the data packets arrival process is Poisson with the rate [1]

$$\lambda_i = \alpha(N-i),\ i=0,1,\ldots,N, \tag{1}$$

where α – data packet arrival rate from an idle call source, i – number of sources in service, N – number of call sources in the system.
The primitive packet flow from finite customer population N arrives to finite buffer of size K system and is served by full availability m channels as shown in Figure 1.

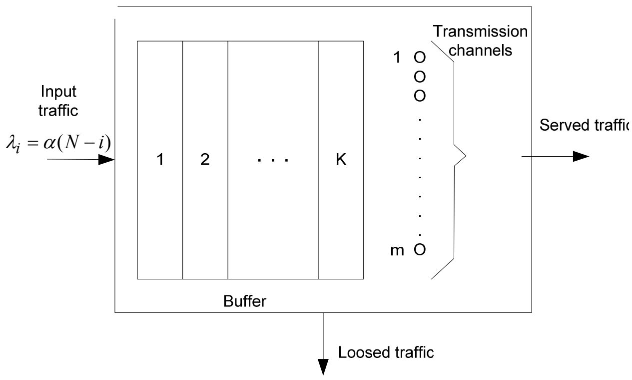

Fig. 1. *M/M/m/K/N* system structure

Consider the data packet transmission time in channel is distributed exponentially with mean value $1/\mu$:

$$P\{T_{trans} \le t\} = 1 - e^{-\mu t},\ 0 \le t < \infty, \mu > 0, \tag{2}$$

where $\mu = 1/\overline{T}_{trans}$ is the data packet transmission rate in the channel, $\overline{T}_{trans}$ - mean value of data packet transmission time in channel.

In this case our system is expressed by Kendall notation like *M/M/m/K/N*, where:

1) First M – represents exponential inter arrival times between packets distribution (Poisson process);
2) Second M – represents exponential data packets transmission times distribution;
3) m – represents number of transmission channels;
4) K – represents max number of data packets in queue (buffer capacity);
5) N – represents number of packet sources.

Our system can be mapped onto Markov process and then mathematically evaluated in terms of this process. Processes in *M/M/m/K/N* system are birth-death process and Markov state transition diagram is shown in Figure 2.

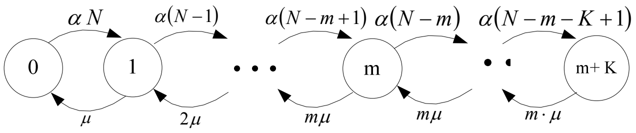

Fig. 2. Markov state transition diagram of *M/M/m/K/N* birth-death system

The number of packets in queue may vary from 0 to the max number K of buffer capacity. If the number of packet sources N equal or less than sum of channel number m and buffer capacity K then we have full availability system without packet looses, otherwise we have the queuing system with packet loss.

Using the global balance concept, we can easily write down the following equations for the system state probabilities evaluation:

$$\begin{cases} \alpha N P_0 = \mu P_1, \\ [\alpha(N-1)+\mu]P_1 = \alpha N P_0 + 2\mu P_2, \\ \ldots\ldots \\ [\alpha(N-m)+m\mu]P_m = \alpha(N-m+1)P_{m-1} + m\mu P_{m+1}, \\ \ldots\ldots \\ m\mu P_{m+K} = \alpha(N-m-K+1), \\ \sum_{i=0}^{m+K} P_i = 1. \end{cases} \quad (3)$$

However, customers arriving to find K already in the system are lost and return immediately to the arriving state as if they had just completed service. This leads to the following set of birth intensities [4]:

$$\lambda_i = \begin{cases} \alpha(N-i), & 0 \le i \le K+m-1, \\ 0, & i > K+m-1, \end{cases} \quad (4)$$

For the death process in system we have the following set of intensities:

$$\mu_i = \begin{cases} i\mu, & 0 \le i \le m, \\ m\mu, & i \ge m. \end{cases} \quad (5)$$

Now we can represent steady state probabilities for the system shown in Figure 2 [4]:

$$P_i = P_0 \prod_{j=0}^{i-1} \frac{\alpha(N-j)}{(j+1)\mu} = P_0 \left(\frac{\alpha}{\mu}\right)^i C_N^i, \; 0 \le i \le m-1, \quad (6)$$

$$P_i = P_0 \prod_{j=0}^{m-1} \frac{\alpha(N-j)}{(j+1)\mu} \prod_{j=m}^{i-1} \frac{\alpha(N-j)}{m\mu} = P_0 \left(\frac{\alpha}{\mu}\right)^i C_N^i \frac{i!}{m!} m^{m-i},$$

$$m \le i \le m+K. \quad (7)$$

The probability of free system P_0 is found from the conservation of probability: $\sum_{i=0}^{m+K} P_i = 1$.

In order to solve these equations (3), we obtain the system steady state probabilities P_i. We now proceed to find other system performance measures such as:

1) Probability of data packet blocking (loss):

$$B = P_{m+K}; \quad (8)$$

2) The probability that data packet will be forced to wait in queue:

$$P\{W > 0\} = \sum_{i=m}^{m+K-1} P_i; \quad (9)$$

3) The mean value of data packets number waiting in the queue:

$$\overline{N}_q = \sum_{i=m+1}^{m+K} (i-m) P_i; \quad (10)$$

4) Waiting position utilization:

$$y_k = \frac{\overline{N}_q}{K}; \quad (11)$$

6) Served traffic intensity or mean value of occupied channels number in the system:

$$Y_{serv} = \sum_{i=1}^{m} i P_i; \quad (12)$$

7) Each channel average utilization:

$$y_m = \frac{Y_{serv}}{m}; \quad (13)$$

8) The average number of customers in the system:

$$\overline{N}_s = \sum_{i=1}^{m+K} i P_i; \quad (14)$$

9) The average data packets arrival rate to the system:

$$\overline{\lambda} = \sum_{i=1}^{m+K} \alpha(N-i) P_i; \quad (15)$$

10) The mean waiting time in queue from the Little formula [2]:

$$\overline{W} = \frac{\overline{N_q}}{\overline{\lambda}}; \qquad (16)$$

11) The mean time spent by packet in the system:

$$\overline{R} = \overline{W} + \overline{T}_{apt}. \qquad (17)$$

Numerical examples of *M/M/m/K/N* system performance measures

The examples of calculated performance measures for the system *M/M/m/K/N* are shown in Figures 3-7.

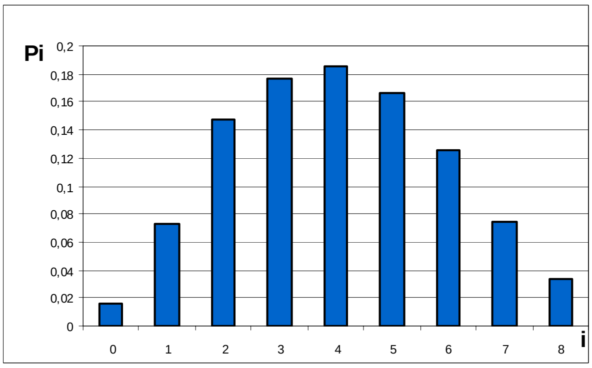

Fig. 3. *M/M/m/K/N* system steady state probabilities P_i distribution, where α=0,45, N=10, μ=1, m=3, K=8

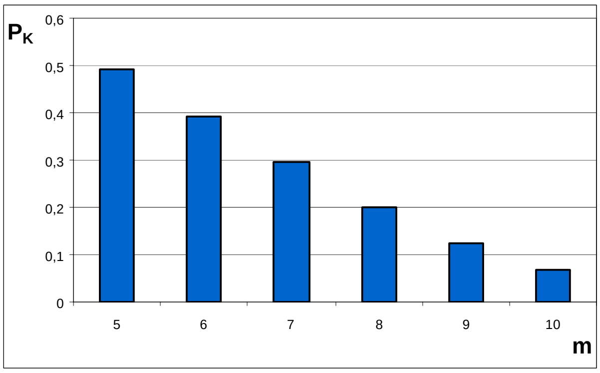

Fig. 4. Data packet loss as a function of channel number m, where α=0,035, N=300, μ=1, K=20

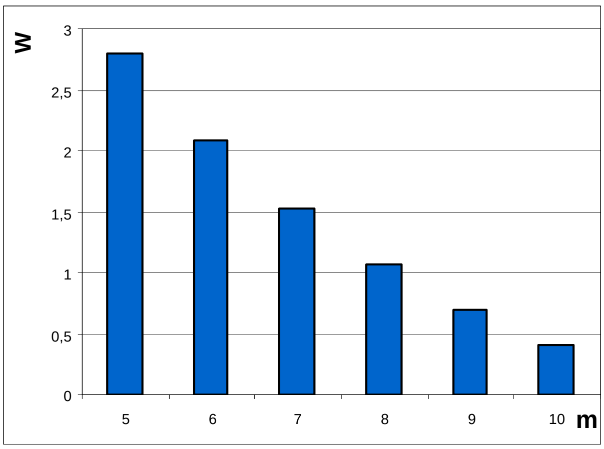

Fig. 5. The mean waiting time in queue W as a function of channel number m, where α=0,035, N=300, μ=1, K=20

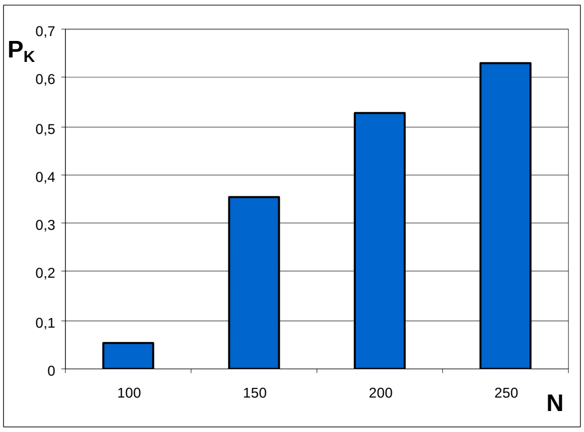

Fig. 6. Data packet loss as a function of customers number N, when α=0,035, m=3, μ=1, K=20

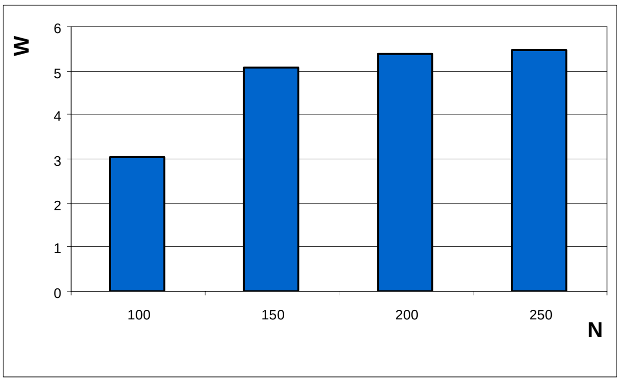

Fig. 7. The mean waiting time in queue W as a function of data packet sources population N, when α=0,035, m=3, μ=1, K=20

Analysis of an asymmetric queuing system *M/M/2/K/N* with finite customer population and finite buffer capacity

This section describes the Markov birth-death model used for calculating characteristics of asymmetric queuing system *M/M/2/K/N* shown in Figure 8, when the fastest free channel is selected. We examine the system linked to two channels with different packets transmission bit rates $\mu_1 > \mu_2$. Analytical model which was proposed in [7] is useful for our system analysis. Data packets arrive according the primitive flow model. Data packet length is exponentially distributed.

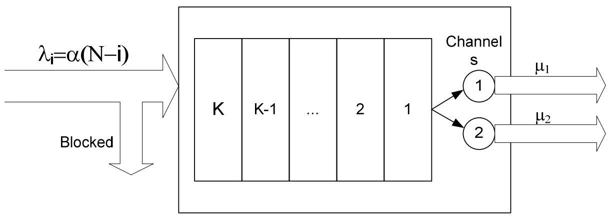

Fig. 8. Architecture of an asymmetric *M/M/2/K/N* system

The finite capacity asymmetric system shown in Figure 8 now we represent by means of continuous Markov chains when the system is in stable state (Fig. 9).

Each system state may be described by vector with three parameters: X,Y,Z, where X represents state of channel 1 (0-free, 1- first channel is transmitting data packet), Y represents state of channel 2, and Z represents number of data packets in the buffer (in our case from 0 to K).

Then, using the global balance concept, we can easily write down the following (18) equations for the system state probabilities P_{XYZ} evaluation.

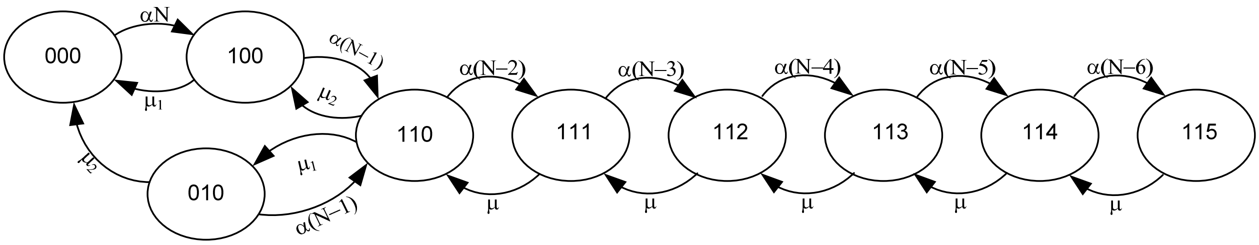

Fig. 9. Markov chains in an asymmetric telecommunication system when the fastest channel is selected first and buffer capacity K=5

$$\begin{cases} \alpha N P_{000} = \mu_1 P_{100} + \mu_2 P_{010}, \\ (\mu_1 + \alpha(N-1))P_{100} = \alpha N P_{000} + \mu_2 P_{110}, \\ (\mu_2 + \alpha(N-1))P_{100} = \mu_1 P_{110}, \\ (\mu + \alpha(N-2))P_{110} = \alpha(N-1)(P_{100} + P_{010}) + \mu P_{111}, \\ (\mu + \alpha(N-3))P_{111} = \alpha(N-2)P_{110} + \mu P_{112}, \\ (\mu + \alpha(N-4))P_{112} = \alpha(N-3)P_{111} + \mu P_{113}, \\ (\mu + \alpha(N-5))P_{113} = \alpha(N-4)P_{112} + \mu P_{114}, \\ (\mu + \alpha(N-6))P_{114} = \alpha(N-5)P_{113} + \mu P_{115}, \\ \mu P_{115} = \alpha(N-6)P_{114}, \\ P_{000} + P_{100} + P_{010} + \sum_{k=0}^{5} P_{11k} = 1, \end{cases} \quad (18)$$

In order to solve these equations, we have to obtain the system state probabilities and then proceed to find such system performance measures:

1. Data packets loss probability:

$$P_{Loss} = P_{11K}. \quad (19)$$

2. First data transmission channel utilization:

$$\rho_1 = P_{100} + P_{110} + \sum_{i=1}^{K} P_{11i}. \quad (20)$$

3. Second data transmission channel utilization:

$$\rho_2 = P_{010} + P_{110} + \sum_{i=1}^{K} P_{11i}. \quad (21)$$

4. Mean value of served traffic intensity:

$$Y = \rho_1 + \rho_2. \quad (22)$$

5. Average number of packets in queue:

$$\overline{N}_q = \sum_{i=1}^{K} i \cdot P_{11i}. \quad (23)$$

6. The effective packet arrival rate into the system:

$$\overline{\lambda} = \alpha \cdot N \cdot P_{000} + \alpha(N-1)[P_{100} + P_{010}] + \alpha(N-2)P_{110} + \\ + \alpha(N-3)P_{111} + \ldots\alpha(N-K+1)P_{11(K-1)} = \alpha \cdot N \cdot P_{000} + \\ + \alpha(N-1)(P_{100} + P_{010}) + \sum_{i=0}^{K-1} \alpha(N-2-i) \cdot P_{11i} \quad (24)$$

7. Average waiting time in queue for the packet:

$$\overline{W} = \frac{\overline{N}_q}{\overline{\lambda}}. \quad (25)$$

8. The probability for packets to wait in buffer:

$$P\{w > 0\} = \sum_{i=0}^{K} P_{11i}. \quad (26)$$

9. Mean number of packets in the system:

$$\overline{N}_s = \overline{N}_q + \rho_1 + \rho_2. \quad (28)$$

10. Average time spent by packet in the system:

$$\overline{T}_s = \frac{\overline{N}_s}{\overline{\lambda}}. \quad (29)$$

Some asymmetric system performance measures examples calculated using an analytical model is shown in Figures 10-15.

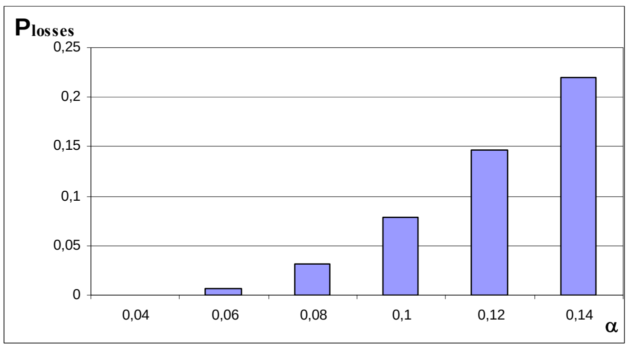

Fig. 10. Packet loss probability as function of packet arrival intensity α, when μ_1=2, μ_2=1, N=30, K=5

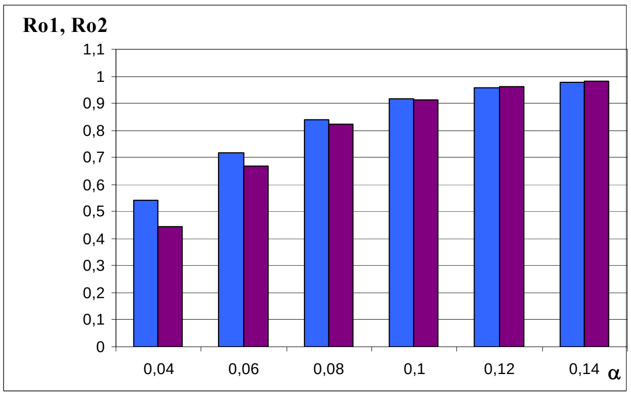

Fig. 11. Channel utilizations as function of packet arrival intensity α, when μ_1=2, μ_2=1, *N*=30, K=5

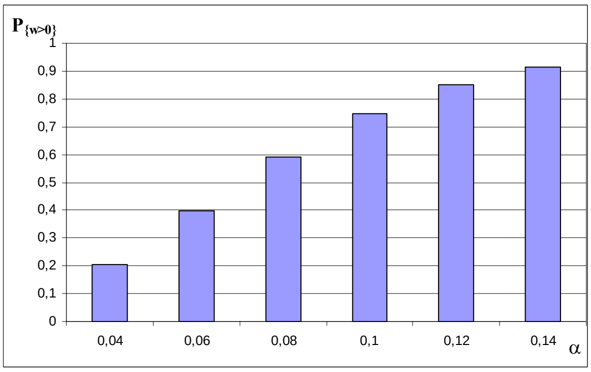

Fig. 12. Probability of packet delay as function of packet arrival intensity α, *N*=30, K=5

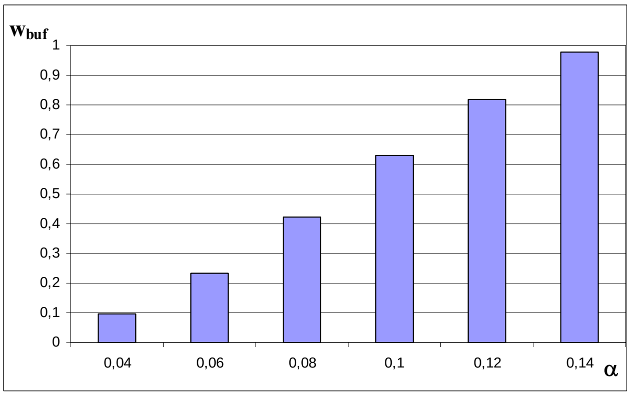

Fig. 13. Mean waiting time in buffer for packet as function of packet arrival intensity α, when μ_1=2, μ_2=1, *N*=30, K=5

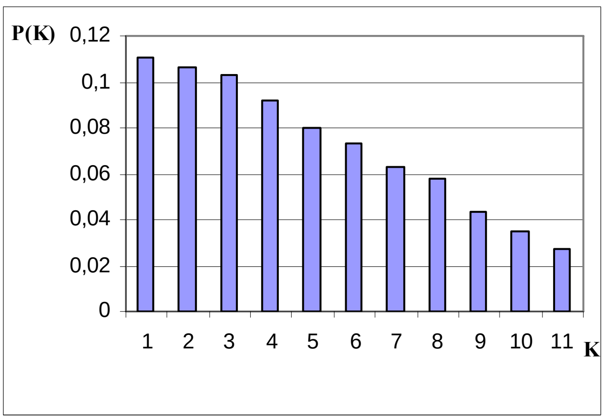

Fig. 14. Packets loss probability ***P***(*K*) as function of the buffer capacity *K*, when μ_1=2, μ_2=1, *N*=50, α=0.06

Using the analytical models from paper sections 1 and 3 we compare both systems performance measures. Packet loss as function of α are shown in figure 15, when both systems throughput and incoming traffic is the same, but symmetric system with three channels and an asymmetric system with two channels are involved.

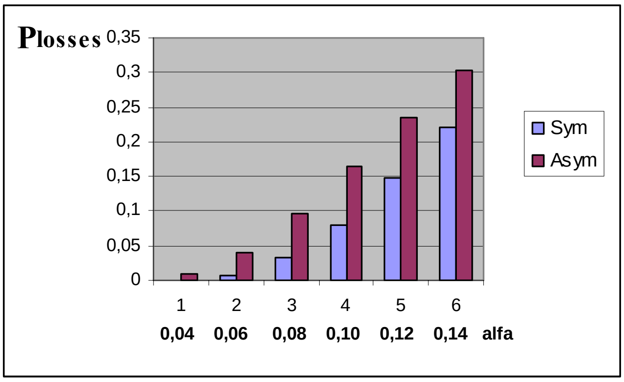

Fig. 15. Packet loss probability as function of packet arrival intensity α, where *N*=30, K=5, Series1 - for symmetric system μ_1=1, μ_2=1, μ_3=1, Series2 – for asymmetric system μ_1=2, μ_2=1

Conclusions

We believe that the offered mathematical models of the systems are useful for analysis processes in data networks. The continuous time Markov chains give possibility for easy and quick analysis of symmetric and the asymmetric system with finite buffer servicing primitive packets flow.

In the asymmetric system with a heavy traffic both channels are usually occupied and therefore a channel selection strategy is not so important and the system may be investigated by means of general symmetric queuing model.

References

1. **Iversen V. B, Sanders B.** Engset formula with continuous parameters – Theory and Applications. AEU, International Journal of Electronics and Communications, Vol. 55, No.1, 2001. – P. 3-9.
2. **Little J.** A Proof of the Queuing Formula L= λW. Operations Research. - May 1961. –No. 9(3). – P. 383-387.
3. **Konheim A., Reiser M.** A Queuing Model with Finite Waiting Room and Blocking. Journal of the ACM.- April 1976 – No. 23(2). – P. 328-341.
4. **Kleinrock L.** Queuing Systems: Problems and Solutions. Wiley, New York, 1996. – 227 p.
5. **Rindzevičius R.** Procesų ribotos eilės sistemose tyrimas // Elektronika ir Elektrotechnika.- Kaunas: Technologija, 2000.- Nr.6(29).- P.22-24.
6. **Pilkauskas V., Rindzevičius R.** Visiško pasiekiamumo linijų pluoštas, aptarnaujantis primityviojo srauto paraiškas, kurių dalis palieka eilę // Elektronika ir Elektrotechnika.- Kaunas: Technologija, 1997.- Nr. 4(13).- P.67-71.
7. **Rindzevicius R., Pilkauskas V., Gvergzdys K. "**Analysis of an Asymmetric Data Network Node with Priority Call Flows", In Proc. of the 27[th] International Conference on Information Technology Interfaces, June 20-23, 2005, Cavtat, Croatia. - P. 513-520.

Presented for publication 2006 01 15

R. Rindzevicius, D. Poskaitis, B. Dekeris. Performance Measures Analysis of *M*/*M*/*m*/*K*/*N* Systems with Finite Customer Population // Electronics and Electrical Engineering. – Kaunas: Technologija, 2006. – Nr. 3(67). – P. 65–70.

This article provides an analytic solution for performance measures of finite channel capacity *m* queuing system *M*/*M*/*m*/*K*/*N*. Queuing system with a finite number of call sources size *N* and restricted storage *K* is investigated. Markov chains are used for the performance measures evaluation. The solution to our queuing system with losses, which is the birth-death system, was obtained by applying the finite Markov chains. There is used system with primitive call flow model. Interarrival time and service time represented using exponential distribution. Packets arriving to find *K*+*m* already in the system are lost and return immediately to the arriving state as if they had just completed service. The most important performance measures of queuing system are defined in an analytic form, such as delay probability *P*(*W*>0), mean value of waiting time mean value of queue length $E[N_q]$, call loss probability *B,* service channel utilization y_m, and etc. Taken queuing models for *M*/*M*/*m*/*K*/*N* systems are very elegant in analysis and provide good closed results. Some of these models were extensively used in designing telephone networks, but however they are very useful in modeling of packets or messages services in data networks. We describe the Markov birth-death model used for calculating characteristics of asymmetric queuing system *M/M/2/K/N*, when the fastest free channel is selected. So we examine the system linked to two channels with different packets transmission rates $\mu_1 > \mu_2$. Some calculation results are presented in diagram form. Ill. 15, bibl. 7 (in English; summaries in English, Russian and Lithuanian).

Р. Риндзявичюс, Д. Пошкайтис, Б. Декерис. Исследование характеристик производительности систем *M*/*M*/*m*/*K*/*N* с конечным числом источников // Электроника и электротехника. – Каунас: Технология, 2006. – № 3(67). – С. 65–70.

Представлены аналитические выражения характеристик работы ограниченной системы с ожиданием *M*/*M*/*m*/*K*/*N*. Анализируется система с ожиданием с конечным числом источников *N* при ограниченной ёмкости *К* буфера. Для оценки характеристик работы системы с ожиданием применены Марковские цепи. Решения *M*/*M*/*m*/*K*/*N* системы рождения и гибели получены при помощи конечной цепи Маркова при обслуживании примитивного потока вызовов. Длительность промежутков между вызовами и продолжительность обслуживания распределены экспоненциально. Вызов, попавший в систему когда в ней находятся *K*+*m* вызовов не обслуживается и немедленно возвращается в свободное состояние так же как и в случае окончания обслуживания вызова. Основные характеристики качества обслуживания вызовов определены в аналитической форме: вероятность ожидания *P*(*W*>0), среднее время ожидания в очереди E[*W*], средняя длина очереди $E[N_q]$, вероятность необслуживания вызова *B*, использования канала обслуживания y_m и др. Разработанные модели систем *M*/*M*/*m*/*K*/*N* очень удобные для их анализа и обеспечивают хорошое представление характеристик работы системы. Часть моделей *M*/*M*/*m*/*K*/*N* применяются для проектирования телефонных сетей, но они также полезные для анализа сетей передачи данных. Марковские цепи рождения и гибели использованы для определения характеристик работы асимметрической системы с двумя каналами различной скорости передачи данных, когда канал с большей скоростью передачи занимается первым. Некоторые результаты расчётов приведены в графиках. Ил. 15, библ. 7 (на английском языке; рефераты на английском, русском и литовском яз.).

R. Rindzevičius, D. Poškaitis, B. Dekeris. *M*/*M*/*m*/*K*/*N* sistemos, esant baigtiniam paraiškų šaltinių skaičiui, našumo rodiklių analizė // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2006. – Nr. 3(67). – P. 65–70.

Pateikiamos baigtinės talpos masinio aptarnavimo sistemos *M*/*M*/*m*/*K*/*N* darbo našumo charakteristikų analitinės išraiškos. Tiriama masinio aptarnavimo sistema, turinti ribotą šaltinių skaičių *N* ir baigtinę buferio talpą *K*. Sistemos darbo našumo rodikliams įvertinti taikomi Markovo procesai. Masinio aptarnavimo sistemos su nuostoliais, kuri pateikta žūties ir dauginimosi grandine, sprendiniai gauti naudojant baigtinę Markovo grandinę. Tirti pasirinktas primityvaus paraiškų srauto modelis. Laikotarpiai tarp paraiškų atėjimo momentų ir paraiškos aptarnavimo trukmė pasiskirstę pagal eksponentinį dėsnį. Į sistemą atėjus vartotojo paraiškai momentu kada joje jau yra *m*+*K* paraiškų, ji nėra aptarnaujama ir vartotojas grąžinamas į laisvą būseną, kaip ir baigus paraiškos aptarnavimą. Gautos sistemos darbą nusakančių charakteristikų analitinės išraiškos: paraiškos vėlinimo tikimybė *P*(*W*>0), vidutinė laukimo eilėje trukmė E[*W*], vidutinis eilės ilgis $E[N_q]$, paraiškų nuostolių dydis *B*, aptarnavimo kanalo naudojimo koeficientas y_m ir kt. Pateikti *M*/*M*/*m*/*K*/*N* sistemų modeliai labai patogūs sistemoms analizuoti ir pateikia glaustus rezultatus. Dalis šių modelių panaudojama telefono tinklams projektuoti, bet taip pat jie taikytini duomenų perdavimo tinklams analizuoti. Asimetrinės masinio aptarnavimo sistemos *M*/*M*/2/*K*/*N* darbo charakteristikoms apskaičiuoti panaudoti Markovo žūties ir dauginimosi procesai, kai pirmasis užimamas didesnės spartos kanalas. Dalis skaičiavimo rezultatų pateikta grafikuose. Il.15, bibl. 7 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).