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Analysis of Uniform Polar Quantization over Bennett's Integral

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Introduction

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Furrier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in polar format (i.e. magnitude and phase components). In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation [1]. Polar quantization consists of separate but uniform magnitude and phase N level quantization, so that rectangular coordinates of the source (x,y) are transformed into the polar coordinates in form: $r=(x^2+y^2)^{1/2}$, $\phi=\tan^{-1}(y/x)$ where r represents magnitude and ϕ is phase. In previous works about polar quantization [1,3] only product uniform quantization was always considered ($\boldsymbol{N}=M\times L$). That optimization approximated granular distortion as:

$$D_g = \frac{r_{\max}^2}{12L^2} + \frac{2\pi^2}{3M^2}. \qquad (1)$$

The optimal uniform polar quantization (OUPQ) is very similar to the original uniform polar quantization (UPQ) except the fact that the number of the regions for the phase angle varies depending on the result of magnitude quantization. In other words, each concentring ring in quantization pattern is allowed to have a different number of partitions in the phase quantizer (P_i) when r is in the i-th magnitude ring. Their implementation remains simple requiring only two scalar quantizers and lookup table of the P_i. One UPQ must satisfy the constraint $\sum_{i=1}^{L} P_i = N$ in order to use all of N regions for the quantization.

In this paper polar quantizers are designed and analysed under additional constraint – scalar quantizer is a uniform one. This restriction has the following advantages over optimal polar quantization: the implementation is simple, and no compressor-expander pair is needed. In [5] is given the analysis of vector quantization in order to determine the optimal maximal amplitude. In [5] is given the analyses for asymptotic uniform polar quantization. Optimisations are done with respect to granular distortion D_g , i.e. the mean-square error (MSE). The goal of this paper is to find simple equation for distortion by solving Bennett's integral for uniform polar quantization and circular symmetric sources (iid Gaussian source). The analysis for optimal uniform polar quantization and optimal product polar quantization is done.

Uniform polar quantization

For this analysis we use uniform polar quantizer with L magnitude levels and P_i phase reconstruction levels at the magnitude reconstruction level m_i, $1\le i\le L$. First we portion the magnitude range $[0,r_{L+1}]$ into magnitude rings with $L+1$ decision levels $\boldsymbol{r}=(r_1,\ldots,r_{L+1})$ and $(0=r_1<r_2<\ldots<r_L<r_{L+1}=r_{\max})$. The magnitude reconstruction levels $\boldsymbol{m}=(m_1,\ldots,m_L)$ obviously satisfy succession $(0<m_1<m_2<\ldots<m_L)$. Let us assume that the total number of reconstruction points N is large enough. In that case magnitude decision levels and reconstruction levels are given in turn:

$$r_i = (i-1)\Delta, \quad 1 \le i \le L+1; \qquad (2)$$

$$m_i = (i-1/2)\Delta, \; 1 \le i \le L. \qquad (3)$$

Let us consider distortion D as a function of the vector $P=(P_i)_{1\le i\le L}$ whose elements are numbers of phase quantization levels at the each magnitude level. Said in other words, each concentric ring in quantization pattern is allowed to have a different number of partitions in the phase quantizer (P_i) when r is in the i-th magnitude ring. Assuming the representation points are centered in their respective cells, magnitude decision levels and reconstruction levels are given as in equations (2) i (3). Next, we make a partition of each magnitude ring into P_i phase subpartitions. By denoting adjacent phase decision levels with $\phi_{i,j}$ and $\phi_{i,j+1}$, and the j-th phase reconstruction level as $\Psi_{i,j}$ for the i-th magnitude ring, $1\le j\le M_i$, following dependence is valid

$$\phi_{i,j} = (j-1)\frac{2\pi}{P_i}. \qquad (4)$$

Total distortion D is a combination of granular and overload distortions, $D=D_g+D_o$. The granular distortion D_g is given by:

$$D_g = \sum_{i=1}^{L}\sum_{j=1}^{P_i}\int_{\phi_{i,j}}^{\phi_{i,j+1}}\int_{r_i}^{r_{i+1}}\left[r^2 + m_i^2 - 2rm_i\cos(\phi-\psi_{i,j})\right]\frac{f(r)}{2\pi}drd\phi \ . \quad (5)$$

For this analysis we assume that the input is from a continuously valued circularly source with unit variance rectangular coordinate margins and bivariate density function:

$$f(x,y) = p(\sqrt{x^2+y^2}) \ . \quad (6)$$

Transforming to polar coordinates, the phase is uniformly distributed on $[0,2\pi)$ and the magnitude is distributed on $[0,\infty)$ with density function $f(r) = 2\pi r p(r)$. The magnitude and phase are independent random variables.

The transformed probability density function for the Gaussian source is:

$$f(r,\phi) = \frac{1}{2\pi\sigma^2}re^{-\frac{r^2}{2\sigma^2}} = \frac{f(r)}{2\pi} \ . \quad (7)$$

Without losing generality we assume that variance is $\sigma^2=1$.

Suppose that a polar quantizer has many points which are small and the source density is smooth. In that case granular distortion D_g of one cell is given by:

$$D_{i,j} = \frac{1}{2}\int_{R_{i,j}}\left[r^2 + m_i^2 - 2rm_i\cos\left(\phi-\psi_{i,j}\right)\right]\frac{f(r)}{2\pi}drd\phi \ , \quad (8)$$

$$D_{i,j} = \frac{1}{2}\frac{f(m_i)}{2\pi}\int_{R_{i,j}}\left[r^2 + m_i^2 - 2rm_i\cos\left(\phi-\psi_{i,j}\right)\right]drd\phi \ . \quad (9)$$

The total granular distortion for polar quantization is found in [4]:

$$D_g^{pol} = \sum_{i=1}^{L}\sum_{j=1}^{P_i}D_{i,j} = \sum_{i=1}^{L}P_iD_{i,j} = \sum_{i=1}^{L}\frac{\Delta}{12}\left(\frac{\Delta^2}{2}+\frac{2m_i^2\pi^2}{P_i^2}\right)f(m_i) \ . \quad (10)$$

Bennett's integral for uniform polar quantization

A two-dimensional N-point scalar quantizer is characterized by a partition $S=\{S_1,S_2,...,S_N\}$ of two-dimensional Euclidean space R^2 into N quantization cells and a code book, noticed as $C=\{\mathbf{y}_1,\mathbf{y}_2,...,\mathbf{y}_N\}$, consisting of N quantization points in two-dimensional Euclidean space R^2. A two-dimensional source vector $\mathbf{x}=\{x_1,x_2\}$ is quantized into one of the $\mathbf{y}_i$'s according to the quantization rule $Q(\mathbf{x})=\mathbf{y}_i$ if $\mathbf{x}\in S_i$. Encoding rate for two-dimensional quantizer is $\log_2(N/2)$. When applied to a random vector $\mathbf{x}=\{x_1,x_2\}$ with probability density function $p(\mathbf{x})$, quantizer's distorsion is given by:

$$D(S,C) = \frac{1}{2}\int\left\|\mathbf{x}-Q(\mathbf{x})\right\|^2 p(\mathbf{x})d\mathbf{x} \ ; \quad (11)$$

$$D(S,C) = \frac{1}{2}\sum_{i=1}^{N}\int_{S_i}\left\|\mathbf{x}-\mathbf{y}_i\right\|^2 p(\mathbf{x})d\mathbf{x} \ ; \quad (12)$$

where $\left\|\mathbf{x}-\mathbf{y}_i\right\|$ denotes Euclidian distance

$$\left\|\mathbf{x}-\mathbf{y}_i\right\| = \left(\sum_{j=1}^{2}\left(x_j-y_{ij}\right)^2\right)^{\frac{1}{2}} \quad (13)$$

and $p(\mathbf{x})$ is the two-dimensional density of $\mathbf{x}$.

Bennett showed that the mean-squared error of a scalar quantizer (k=1), with many small cells (N large) and with each y_i in the center of its cell may be accurately approximated as:

$$D(S,C) \cong \frac{1}{12N^2}\int\frac{1}{\lambda(x)^2}p(x)dx \ , \quad (14)$$

where $\lambda(x)$ is a function, called point density, and $\lambda(x)\Delta$ is the fraction of quantization points in a small interval of width Δ surrounding x. The integral without limits denotes an integral over the entire space. The right-hand side of previous equation is known as Bennett's integral. Although originally derived for companders (quantizers consisting of a compressor mapping, uniform quantizer, and expander mapping) with λ equal to the derivative of the compressor function can be recognized by other quantizers, and is applied more generally. Bennett's integral shows how the distortion depends on the key characteristics of the quantizer, namely, the number of points N and a point density λ. Its utility is exemplified by the fact that one may use it to show that the best quantizers have

$$\lambda(x) \cong \frac{p(x)^{1/3}}{\int p(x)^{1/3}dx} \ . \quad (15)$$

In following analysis we extend Bennett's integral to polar quantizer. The goal is to give simple approximate formula for distortion that shows influence of key characteristics.

Generally, the normalized moment of inertia is defined as:

$$NM(i) = \frac{1}{k}\frac{1}{vol^{(1+2/k)}(S_i)}\int_{S_i}\left\|\mathbf{x}-\mathbf{y}_i\right\|^2 d\mathbf{x} \ . \quad (16)$$

In our case, for two-dimensional N-point scalar quantization, the above equations become:

$$D(S,C) = \sum_{i=1}^{L}\sum_{j=1}^{P_i}p\left(m_i,\psi_{i,j}\right)\cdot NM\left(i\right)\cdot vol^2\left(S_{i,j}\right), \quad (17)$$

where $vol(S_{i,j})$ denotes the volume of the cell $S_{i,j}$ and $NM(i)$ denotes the normalized moment of inertia of the cell $S_{i,j}$ around the point y_i with respect to the distortion and can be expressed as

$$vol(S_{i,j}) = rdrd\varphi = \frac{(r_{i+1}^2-r_i^2)\pi}{P_i} = \frac{2m_i\Delta_r\pi}{P_i} \ , \quad (18)$$

$$NM(i) = \frac{1}{2}\frac{1}{vol^2(S_{i,j})}\int_{S_{i,j}}\left(\left(x-x_i\right)^2+\left(y-y_i\right)^2\right)dxdy \ . \quad (19)$$

After transformation to polar coordinates and integration over ϕ, the equation above can be written as

$$NM(i)=\frac{1}{2}\frac{1}{vol^2(S_{i,j})}\frac{2\pi}{P_i}\int_{r_i}^{r_{i+1}}[(r-m_i)^2+\frac{rm_i}{3}\frac{\pi^2}{P_i^2}]rdr\ ; \quad (20)$$

$$NM(i)=\frac{1}{12}\left(\frac{P_i\Delta_r}{4m_i\pi}+\frac{m_i\pi}{\Delta_r P_i}\right)\approx\frac{1}{12}\left(\frac{p(r)\Delta_r}{4r\pi}+\frac{r\pi}{\Delta_r p(r)}\right)=m(r,\phi)\ , \quad (21)$$

where Δ_r - the width of rings in the case of restricted uniform polar quantization: $\Delta_r = r_{\max}/L$.

Finally, point density is found as:

$$\lambda(m_i,\psi_{i,j})=\frac{1}{N\cdot vol(S_{i,j})}\approx\frac{1}{N}\frac{p(r)L}{r_{\max}r2\pi}=\lambda(r,\phi)\ . \quad (22)$$

After this approximation, distortion D expressed over Bennett's integral becomes:

$$D(S,C)=\frac{1}{N}\int_0^{r_{\max}}\int_0^{2\pi}p_r(r,\varphi)\frac{m(r,\phi)}{\lambda(r,\phi)}rdrd\phi\ , \quad (23)$$

$$D(S,C)=\int_0^{r_{\max}}re^{-\frac{r^2}{2}}\left[\frac{1}{24}\frac{r_{\max}^2}{L^2}+\frac{1}{6}\frac{r^2\pi^2}{p^2(r)}\right]dr\ . \quad (24)$$

Solving of Bennett's integral for optimal uniform polar quantization

Using the method of Lagrange multipliers with restriction for the total number of the reconstruction points N we obtained optimal point density $p(r)$ as

$$p(r)=\frac{Nr_{\max}}{3L\left(1-e^{-\frac{r_{\max}^2}{6}}\right)}re^{-\frac{r^2}{6}}\ . \quad (25)$$

The distortion can be determined as:

$$D(S,C)=I_1+I_2\ , \quad (26)$$

$$I_1=\int_0^{r_{\max}}\frac{r_{\max}^2}{24L^2}re^{-\frac{r^2}{2}}dr=$$

$$=\frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2\left(1-e^{-\frac{r_{\max}^2}{2}}\right)\approx\frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2\ , \quad (27)$$

$$I_2=\int_0^{r_{\max}}\frac{\pi^2}{6\left(p(r)\right)^2}r^3e^{-\frac{r^2}{2}}dr=$$

$$=\frac{1}{2}\left[3\pi\left(1-e^{-\frac{r_{\max}^2}{6}}\right)\frac{L}{Nr_{\max}}\right]^2\left(1-e^{-\frac{r_{\max}^2}{6}}\right), \quad (28)$$

Finally we find distortion as a solution of Bennett's integral for uniform polar quantizer as:

$$D(S,C)=\frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2+\frac{1}{2}\left[3\pi\left(1-e^{-\frac{r_{\max}^2}{6}}\right)\frac{L}{Nr_{\max}}\right]^2\left(1-e^{-\frac{r_{\max}^2}{6}}\right). \quad (29)$$

We obtain the optimal number of levels after minimizing distortion D over L $(\partial D/\partial L=0)$:

$$L=\frac{r_{\max}}{\sqrt[4]{108\left(1-e^{-r_{\max}^2/6}\right)^3}}\sqrt{\frac{N}{\pi}} \quad (30)$$

and we find distortion:

$$D_g^{opt}=\frac{1}{2}\frac{\pi\sqrt{3}}{N}\left(1-e^{-\frac{r_{\max}^2}{6}}\right)^{\frac{3}{2}}; \quad (31)$$

$$D=D_g^{opt}+D_{over}\ . \quad (32)$$

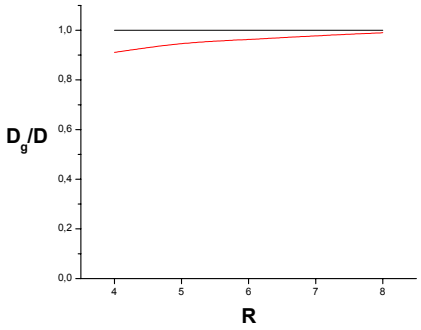

Fig. 1. Relation between granular distortion D_g and total distortion D, for different values of R, case of OUPQ

From Fig.1 we can make a conclusion that for the values $R \geq 7$ the overload distortion D_{over} can be disregarded.

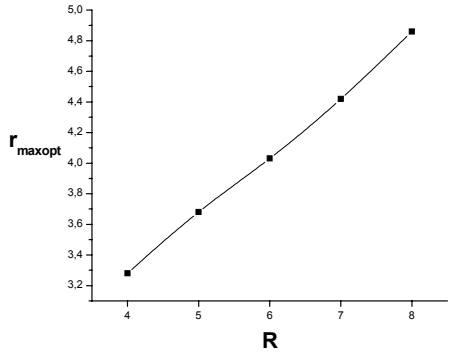

Fig. 2. In addition to process of determination the optimal value of r_{max} ,case of OUPQ

Solving of Bennett's integral for product uniform polar quantization

For product uniform polar quantization:

$$p(r)=p(m_i)=const=\frac{N}{L}=M\ ; \quad (33)$$

$$D(S,C)=\int_0^{r_{\max}} re^{-\frac{r^2}{2}}\left[\frac{1}{24}\frac{r_{\max}^2}{N^2}M^2+\frac{1}{6}\frac{r^2\pi^2}{M^2}\right]dr\,; \quad (34)$$

$$D(S,C)=\frac{1}{24}\frac{r_{\max}^2}{N^2}M^2\cdot P+\frac{1}{6}\frac{\pi^2}{M^2}\cdot I\,, \quad (35)$$

where

$$P=\int_o^{r_{\max}} re^{-\frac{r^2}{2}}dr\approx 1\,; \quad I=\int_0^{r_{\max}} r^3e^{-\frac{r^2}{2}}dr\approx 2\,. \quad (36)$$

After minimizing equation (10) over M

$$\frac{\partial D(S,C)}{\partial M}=0\,, \quad (37)$$

we obtain optimal values of *M* and *L*:

$$M=\sqrt[4]{\frac{8\pi^2N^2}{r_{\max}^2}}\,; \quad L=\sqrt[4]{\frac{r_{\max}^2N^2}{8\pi^2}}\,. \quad (38)$$

The optimal point density for product uniform polar quantization is found as:

$$\lambda^{prod}(r,\phi)=\frac{1}{2\pi r_{\max}r}\,. \quad (39)$$

We find the granular distortion as

$$D_g^{prod}=\frac{r_{\max}\pi\sqrt{2}}{6N}\,. \quad (40)$$

Finally, total distortion for product uniform polar quantization is:

$$D^{prod}=D_g^{prod}+D_{over}\,. \quad (41)$$

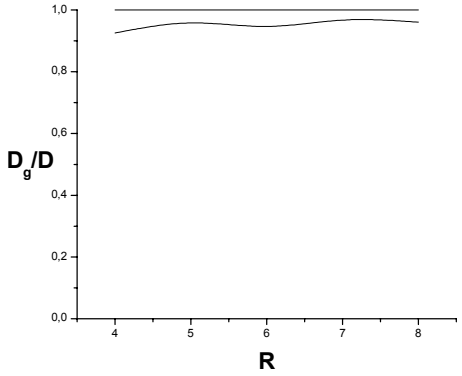

Fig. 3. Relation between granular distortion D_g and total distortion D, for different values of R, case of PUPQ

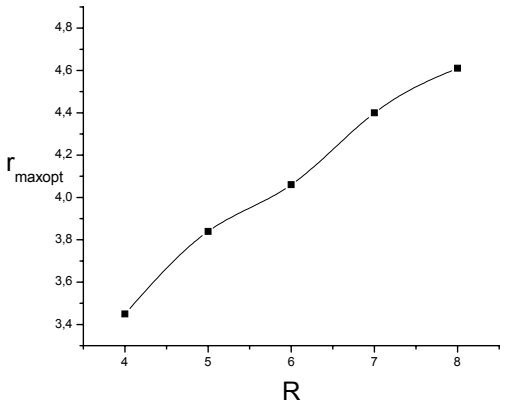

Fig. 4. In adition to process of determination the optimal value of r_{max} ,case of PUPQ

Table 1. Values of Distortion for different values of R, in cases of OUPQ and PUPQ

R	N	D^{opt}	D^{prod}
4	256	0.00808795	0.010818250
5	1024	0.00225094	0.002897594
6	4096	0.00060305	0.000775516
7	16384	0.00015655	0.000205543
8	65536	0.00004031	0.000054328

Conclusion

The analysis of Bennett's integral is given for uniform polar quantization for two-dimensional memoryless Gaussian sources. This paper gives simple and complete analysis for constructing an optimal uniform polar quantizer for sources with optimal point density. We calculated granular distortion and found gain obtained by using optimal point density. The goal of this paper is solving quantization problems for uniform polar quantizers by finding minimal distortion and optimal point density.

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D. A. Stojanović, D. R. Aleksić, Z. H. Perić, A. Z. Jovanović. Tolygaus polinio kvantavimo, naudojant Beneto integralą, tyrimas // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2005. – Nr. 8(64). – P. 5–9.

Analizuojamas Beneto integralas, skirtas tolygiam poliniam kvantavimui dvimačiuose neįsimenančiuose Gauso šaltiniuose, atsižvelgiant į granuliuotą iškraipymą (deformaciją) D_g ir vidutinę standartinę paklaidą. Pateikiamas tikslas – atrasti paprastą iškraipymo lygtį sprendžiant tolygaus polinio kvantavimo Beneto integralą (iid gausiniai šaltiniai). Kvantavimo metodo analizės metu panaudotas variantiškumas σ^2=1. Atlikta optimalaus tolygaus polinio kvantavimo ir optimalaus gaminio analizė. Il. 4, bibl. 5 (anglų kalba; santraukos lietuvių, anglų ir rusų k.).

D. A. Stojanović, D. R. Aleksić, Z. H. Perić, A. Z. Jovanović. Analysis of Uniform Polar Quantization over Bennett's Integral // Electronics and Electrical Engineering. – Kaunas: Technologija, 2005. – No. 8(64). – P. 5–9.

In this paper the analysis of Bennett's integral is given for uniform polar quantization for two-dimensional memoryless Gaussian sources with respect to granular distortion D_g , i.e. the mean-square error (MSE). The goal of this paper is to find simple equation for distortion by solving Bennett's integral for uniform polar quantization and circular symmetric sources (iid Gaussian source). During the quantization method analysis we used variance σ^2=1. The analysis for optimal uniform polar quantization and optimal product polar quantization is done. Ill. 4, bibl. 5 (in English; summaries in Lithuanian, English and Russian).

Д. А. Стоянович, Д. Р. Алексич, З. Г. Перич, А. З. Иованович, Анализ равномерного полярного квантирования при помощи интеграла Беннета // Электроника и электротехника. Каунас: Технология, 2005. – № 8(64). – С. 5–9.

Анализируется интеграл Беннета, используемый при квантировании равномерных негауссовых сигналов. Определено среднеквадратическая погрешность при решении этих задач, когда σ^2=1. Найдены оптимальные условия анализа для равномерной полярной дискретизации параметров продукции. Ил.4, библ.5 (на английском языке; рефераты на литовском, английском и русском яз.).