

Synthesis of Nyquist's Filters in the Time Domain

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Introduction

The synthesis of filters can be carried out in both the frequency and time domain. Commonly, filters are devised in radio engineering to receive signals in a certain frequency range. Therefore methods for the synthesis of filters in the amplitude-frequency plane evolved. In data transmission besides the requirements of frequency range also the requirements of phases emerged, because only the combined characteristic of a filter fully describes the filter as a linear system. Such complete characteristics are also transition processes, which are generated by standard input signals: δ - impulse or Heaviside step signal, but in the discrete data transmission the filters response to unit pulse is equally important. Their advantage is that any of them is real and sufficient to fully describe the filter as a linear system. That creates advantages for the synthesis of filters after transition processes in the time domain. That is an approximation problem in its mathematical nature. We need to know the target functions that must be approximated and the basis with which we must carry out the approximation to solve it.

Nyquist filters of the raised cosine type are common. They have several variations with certain impulse responses as target functions (response to δ -impulse), such as the one that author calls "the generalized" [1] and the traditional form that is being taught in the schools:

$$p(t) = \text{sinc}\left(\frac{t}{T_s}\right) \frac{\cos(\pi \cdot \alpha \cdot t/T_s)}{1 - 4 \cdot \alpha^2 \cdot (t/T_s)^2}, \quad (1)$$

where $\text{sinc} = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$; T_s – symbol length; α – a value

in the interval $[0; 1]$, that describes the filters frequency range, and in the time domain – its oscillation damping rate. The α - quotient is commonly called the roll-off factor in the technical literature [2].

In the analog filter technology filters with their impulse responses equal to function (1) shouldn't be used, as it is impossible to realize the δ -impulses at the input. Approximations of δ - impulse are short pulses with great amplitude, which also cause a response from the filter that comes close to the ideal response (1). But the literature on this topic mostly deals with such filters and presents the analysis of errors, which emerge when activating these filters with short pulses [3].

Therefore it is necessary to create filters the responses of which meet the demands of (1), but are easier to activate with input pulses that are practically more easily realizable. In the frequency domain a solution has been found for rectangular input impulses the length of which (T) is equal to the length of the symbol that must be transmitted (the time limit for that). This idea is based on the series of two filters: one of them is a common Nyquist's filter with the combined frequency response $N_y(\omega)$, but the other is inverse (reciprocal) to the rectangular impulse spectrum. It means that we must create a filter the frequency response of that is a product of these two functions:

$$F_{fin}(\omega) = \frac{T_p \omega}{2 \sin(\frac{1}{2} T_p \omega)} N_y(\omega), \quad (2)$$

where T_p is the length of the rectangular pulse. Formula (2) is the target function for the synthesis of a filter in the space with frame of references (Amplitude, frequency, phase). Such filters are already manufactured (see the oscillogram of a filter with a roll-off factor of $\alpha=0,5$, that's manufactured by *Linear Technology Ltd.*)

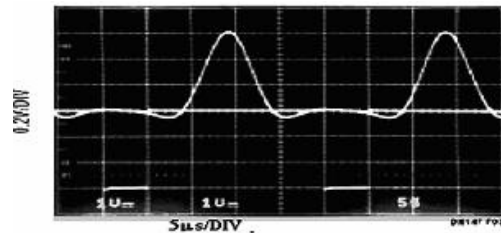


Fig. 1. The pulse response of the LTC1069-7 is fully symmetrical

At the bottom of the oscillogram we can see the rectangular input impulses. The filter's response after each input impulse begins with a negative wave. The lower the roll-off factor is, the more pre- and post-response oscillations occur and the greater is the delay between the maxims of impulse and response, and the more reactive elements are necessary in the circuit. Therefore the circuit becomes more complicated and more expensive, and the synthesis becomes more laborious. "*Nuhertz Technologies L.L.C.*" demonstrates a similar equivalent circuit of a filter, that let's us see the filter's complexity and the very high requirements on the accuracy (tolerance) of reactive

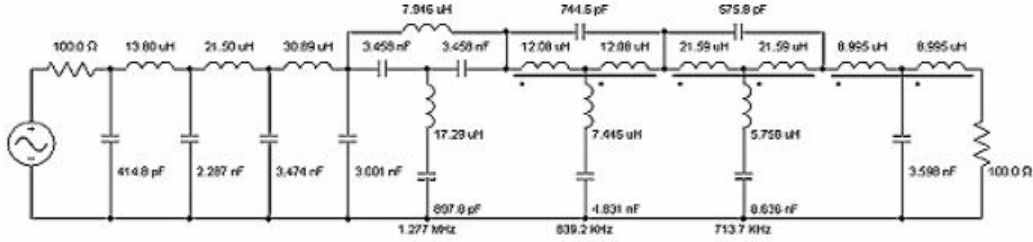


Fig. 2. 7th order raised cosine

elements. Therefore, the circuit is difficult to implement, furthermore it has few identical elements. That causes the growth of expenses. The methods of synthesis in the frequency domain that involve delay circuits (the left side of the scheme) as well as phase driver circuits (the middle of scheme) with a bridge of circuits, which are shunted by series of circuits, lead to such results. As a result the synthesized filter is a passive one with no active elements in it.

The Characteristic Transition Processes of Filters in the Time Domain

When $\alpha=0$ the formula (1) is reduced to the sinc-function.

$$p(t) = \text{sinc}\left(\frac{t}{T_s}\right), \quad (3)$$

which is the theoretical and practically unachievable limit, at which the transmission of data could be carried out in the shortest frequency range - the Nyquist range at a rate of 2 symbols per /sec/Hz.

Summing up the shifted sinc-functions

$$H = \sum_{k=0}^{\infty} \frac{\sin(\pi(x-k))}{\pi(x-k)}, \quad (4)$$

we get a compact sum:

$$H = \frac{1}{2} \frac{\sin(\pi x) \left(\Psi\left(-\frac{x}{2}\right) - \Psi\left(\frac{1-x}{2}\right) \right)}{\pi}, \quad (5)$$

where $\Psi(x) = \text{diff}(\text{GAMMA}(x)) / \text{GAMMA}(x)$. This is the filter' step response for $T_p = T_s = 1$.

The curve of this function likely that of the sinc-function in the interval $(-\infty; -1]$ is crossing the horizontal axis with a regular step = 1, but in the interval $[0; \infty)$ it crosses the horizontal line with the same step at $y=1$.

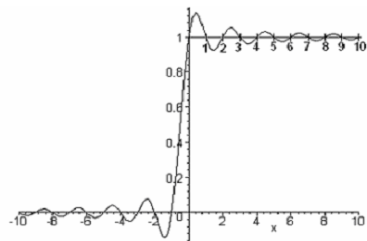


Fig. 3. The transient response of filter for $\alpha=0$

This shape of filter's step response crossing the axis and line (Fig. 3) prevents the interference of symbols.

The pulse response of such a filter, that is the derivation of formula (6)

$$h(x) = \left(-\frac{1}{2} \Psi\left(\frac{1}{2} - \frac{1}{2}x\right) + \frac{1}{2} \Psi\left(-\frac{1}{2}x\right) \right) \cos(\pi x) + \frac{1}{\pi} \times \left(\frac{1}{4} \left(\Psi\left(1, \frac{1}{2} - \frac{1}{2}x\right) - \Psi\left(1, -\frac{1}{2}x\right) \right) \right) \sin(\pi x). \quad (6)$$

differs from the function (1) in it's characteristics of crossing the axis of time. But such a response to the δ -pulse conforms with the response to the unit pulse, the width of which is equal to the length of the symbol that must be transmitted.

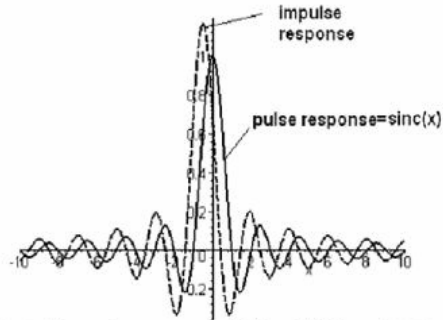


Fig. 4. The pulse response of Nyquist filters for $\alpha=0$

Both the response to the unit pulse and the response to the δ -pulse cannot be realized in praxis. Both of them are damping too slowly in both directions, so we're not able to say that they have a beginning moment, where a trigger for this response could be.

Transition Process Models For Input Pulse With Finite Length In Praxis

Like in the case of the sinc-function the sum by the example (4) can be made from finite number of Nyquist functions. Since finite functions are damping rapidly there is no need to make the sum for infinite number of shifted functions.

Taking the raised cosine function with the roll-off factor = 0,5 it's enough with 10-20 values making up the

sum, to get a stable picture of the transition process (Fig.5), because of rapidly damped oscillations of this function.

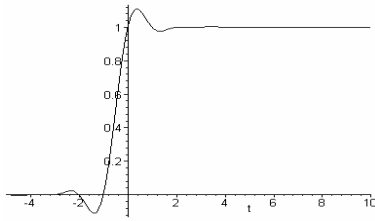


Fig. 5. The sum of 20 raised cosine functions

In the Fig. 5 we can see that the transition process fits in well in the time interval of $[-3; 2]$. So, judging by the transition characteristic the duration of the time period is ~ 5 units of time. But, if the estimation is done, from the impulse response it is shown that the interval of time necessary for the transition process is longer. That might give hope that the result could be more precise when synthesizing for the approximation characteristic. One thing to keep in mind, though, is that the number of components necessary for the synthesis of the filter will grow proportionally to the length of the transition process taken into consideration.

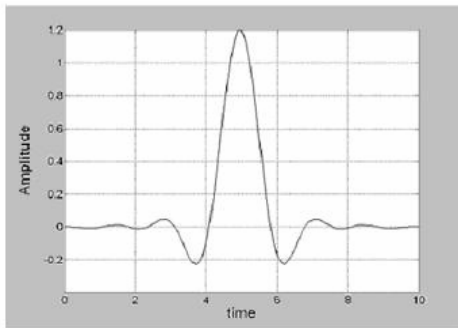


Fig. 6. Impulse response of Nyquist filter built for the square pulses of the unit length

In principle, three patterns can be used for the synthesis of a filter: 1) step response that is the reaction on the Heaviside step, 2) response to the pulse, the length of which is T_s , - it means the building of basis function (a raised cosine function in this case) and 3) impulse response – response (reaction) to a very short pulse with a great amplitude.

The Basic Circuits for the Synthesis of a Filter

The idea of approximation of a function can be used for the synthesis of a filter. If we have picked N four-pole networks, the pulse responses of which are linearly independent, then it is possible to approximate the pulse response of the Nyquist filter $h_N(t)$ by the pulse responses of the basic circuits $\varphi_i(t)$. The same approach can be applied to step responses or responses to a pulse with a definite shape. Basis properties determine the possible accuracy of approximation. An important prerequisite for

the synthesis of filters is that the basis functions must be physically realizable via linear circuits. The responses of these circuits to the input signal will provide the necessary base for approximation. In praxis a finite (limited) number of base links is chosen from a theoretically infinite number of them, which are generated following a determined pattern. We can find the approximation as a weighted sum

$$h_{Ny}(t) \approx \sum_{i=1}^N b_i \varphi_i(t). \quad (7)$$

A filter made by using of this principle will contain an adder and thus it will be an active filter. The advantage of an active filter is that it can be adjusted by changing the b values of the quotients that means it can be adaptive.

The weight quotients b_i must be calculated so that the approximation error could be reduced. This function (basis) can also be carried out with an adaptive algorithm. Although this operation is always possible, the accuracy of the result cannot be constantly guaranteed. The basis must conform with the approximation of the target function and the approximation itself must be with low sensitivity to tolerance of parameters components. Realizing the basic circuits, their pulse responses will be extracted by measuring. This means that each of $\varphi_i(t)$ functions must be treated as a stochastic function that equals only approximately to the response $\phi_{0i}(t)$. That is why the synthesized result $h_{Ny}(t)$ will be a stochastic function too. However the error of the equation won't be significant, if the weight quotients b_i for these real stochastic functions $\varphi_i(t)$ will be precisely calculated and realized. That was tested experimentally. In praxis it is neither possible to measure the $\varphi_i(t)$ precisely, nor to realize the weight quotients b_i as calculated. Using Pugachov's canonic expansion for stochastic functions [4] by theoretical functions, we can write that:

$$D_h(t) = D_b \sum_{k=1}^N [\phi_k(t)]^2, \quad (8)$$

where D_b is the weight quotient's dispersion of deflection from the optimal values. It is easier to calculate the error's energy in the selected approximation interval as an universal indication, than to calculate the dispersion for each point in time

$$E = \int_0^{t_0} D_h(x) dx = D_b \sum_{k=1}^N \int_0^{t_0} [\phi_k(x)]^2 dx. \quad (9)$$

After the transformations we get (8)

$$E = D_b \text{Trace}(\Phi), \quad (10)$$

where Φ is Gram matrix with the following elements:

$$\phi_{n,k} = \int_0^{t_0} \phi_n(x) \phi_k(x) dx, \quad (11)$$

but we must keep in mind that in comparison to article [5] the matrices Φ and Φ^{-1} have changed roles. If the signal is constructed by summing up the orthogonally jointed functions, then the filters are synthesized by summing up the weighed basis functions, which is why in the formula (6) there is a normal matrix instead of the inverse one Φ^{-1} .

The formula (10) cannot be used for precise calculations, because it demands us to know precise basis functions, but it has some importance when we compare different bases. The trace of matrix (Φ) does not depend on the synthesized function, it is determined only by the choice of basis, therefore the lesser the trace (Φ) the better is the selected basis.

The selection of boundaries is important, on this depend the convergence of integrals and the trace of Gram matrix for the biorthogonal set of functions. The next step is the calculation of the inverse matrix $\Psi=\Phi^{-1}$ and the respective orthogonal functions:

$$\psi_k(t) = \sum_{i=1}^N \psi_{k,i} \phi_i(t), \quad (12)$$

which gives the opportunity to calculate the weight quotients

$$b_k = \int_0^{t_0} h_{Ny}(x) \psi_k(x) dx. \quad (13)$$

Trace is not a sufficient criteria, it doesn't mean that the selected family of physically realizable functions is suited for the approximation of the given Nyquist function. Besides the stochastic errors, the systematic errors as characterized by Bessel's inequality also do exist. In the case of choice among biorthogonal functions Bessel's inequality has another variation

$$\int_0^{t_0} h_{Ny}^2(t) dt \geq \sum_{m=1}^N a_m b_m, \quad (14)$$

Where a_m and b_m are the expansion quotients by the basis and the families of functions that are biorthogonal to them. The systematic error is the difference between both sides of the inequality.

$$E_{sys} = \int_0^{t_0} h_{Ny}^2(t) dt - \sum_{m=1}^N a_m b_m. \quad (15)$$

Conclusion

Filters created on the basis of series-connected links are similar in their qualities to digital filters, that are based on the operators of delay. The summing up of all responses with the weight quotients lets us change the quotients thus making the filter adaptive. The synthesis can be carried out using the responses of the links of the basis circuit to the standard signals: 1) Dirac's delta impulse; 2) unit step (Heaviside function); 3) rectangular impulse with the length of the filter's tact. The choice is determined by the trace of the Gram matrix. The lesser the trace, the lesser the sensitivity of the filter to the scattering of parameters.

References

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E. Hermanis. Naikvisto filtrų sintezė laiko būvyje // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2005. – Nr. 8(64). – P. 10–13.

Pateikiama Naikvisto filtrų sintezės laiko būvyje analizė, pristatant laikines charakteristikas su bazinių funkcijų eilėmis. Sintezė yra sukurta tiems filtrams, kurie reaguoja į bendrą impulsą su Naikvisto reakcija. Filtrų sintezei naudojama pagrindinės grandinės eilė. Už jos ryšį su įėjimo signalu atsako linijskai (tiesiogiai) nepriklausomų žingsnių procesas. Bazės tinkamumas yra vertinamas šiais dviem kriterijais: 1) Beselio nelygybe biortogonaliosioms funkcijoms; 2) Gramo matricos pėdsaku, apibūdinančiu filtro jautrumą parametru sklaidai. Il. 6, bibl. 5 (anglų kalba; santraukos lietuvių, anglų ir rusų k.).

E. Hermanis. Synthesis of Nyquist's Filters in the Time Domain // Electronics and Electrical Engineering. – Kaunas: Technologija, 2005. – No. 8(64). – P. 10–13.

This research paper deals with the synthesis of Nyquist's filters in the time domain presenting the filter's time characteristics with a series of basis functions. This synthesis is devised for filters that respond to the unit pulse with a Nyquist's reaction. For the synthesis of a filter in the time domain a base circuit series is used. The responses of it's links to the input signal are lineally independent step processes. The suitability of a base is evaluated by these two criteria: 1) Bessel's inequality for the biorthogonal functions, 2) The trace of the Gram matrix, that characterizes the sensitivity of a filter to the scattering of parameters. Ill. 6, bibl. 5 (in English; summaries in Lithuanian, English and Russian).

Э. Херманис. Синтез фильтров Найквиста во времени // Электроника и электротехника. – Каунас: Технология, 2005. – № 8(64). – С. 10–13.

Предлагается новый подход анализа фильтров Найквиста, когда синтез сигнала осуществляется линейнообразно независимо от шагов квантирования. Обоснование доказано функциями Бесселя и матрицами Грамма. Ил. 6, библи. 5 (на английском языке; рефераты на литовском, английском и русском яз.).