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Critical Analysis of Uncertainty Relations Based on Signal Duration and Spectrum Width

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Introduction

The Fourier Uncertainty Principle [1,2] (FUP) states minimal product of signal spreading in time domain and frequency domain

$$\Delta t \ \Delta\omega \geq 1/2 . \tag{1}$$

This type of uncertainty relation has been rather forgotten in signal processing community. In contrast to this, the coordinate-momentum Heisenberg's Uncertainty Relation $\Delta x \Delta p \geq \hbar/2$ that is mathematically equivalent to the FUP [3,4], has become an extremely important cornerstone for quantum mechanics and even for modern understanding of universe. The reason why FUP has not found wide use in signal processing is the fact that conventional mathematical calculation of signal deviation in frequency domain $\Delta\omega$ must include both positive and negative values of frequency. This yields large $\Delta\omega$ values which are meaningless for most practical applications that consider spectra at positive frequencies (if needed, the omitted half-spectrum for negative frequencies may be restored from symmetry principles). Situation is different in quantum mechanics where the standing waves in potential wells may be described as a sum of two waves moving in opposite directions and thereby inclusion of both positive and negative values of momentum is justified.

Probably the most noticeable (and perhaps only) author who has carefully discussed the signal duration and spectrum width based uncertainty relations for signal processing is A.A. Kharkevich [5]. In chapter 12 of [5] he applied at first different practical criteria for Δt and Δf definition ($f=\omega/2\pi$). Those practical criteria were, e.g., time duration including 90% of pulse energy or spectrum width from f=0 up to first zero in spectrum. He compared results for 5 most common pulse forms. Those results, however, lacked the generality that gives mathematically rigorous calculation of variances and standard deviations [6,7]. So Kharkevich developed also a generalized approach based on "function moments" using ideas of inertia radius calculation from mechanics. At that for evaluation $\Delta\omega$ only positive frequency values were used. Probably already Kharkevich recognized the problem of handling the negative frequencies that come out from the conventional complex Fourier transform. We analyzed the "function moments" approach of Kharkevich and concluded that it is equivalent to FUP (with conventional calculation of variances and deviations) if only positive frequency values are considered.

Below in the present work we will also offer another universal formulation of uncertainty relation for the "train of pulses" case. This uncertainty relation will state that there exist a certain limit value for product of pulse train duration and the spectrum peak width for every harmonic of signal. The respective Δt and $\Delta\omega$ definitions are explained below in Fig. 1.

The used definitions

Here we use conventional formulation of Fourier transform $f(t) \leftrightarrow F(\omega)$ for the aperiodic signal:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i\omega\ t)\ f(t)dt , \tag{2}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i\omega\ t)\ F(\omega)dt . \tag{3}$$

(2) defines the following symmetry rules: symmetric component of signal f_S(t)=[$f(t)$+$f(-t)$]/2 creates real component Re[$F(\omega)$] of spectrum that is also symmetric in frequency domain; the antisymmetric component of signal f_{AS}(t)=[$f(t)$-$f(-t)$]/2 creates imaginary component of spectrum Im[$F(\omega)$] that is antisymmetric in frequency domain.

The conventional calculation of standard deviations through the square roots of variances [6, 7] uses signal and its spectrum "energy densities" expressed by module squares f^2 and F^2:

$$\Delta t = \sqrt{Var(t)}\,, \quad (4)$$

$$Var(t) = \int_{-\infty}^{+\infty} (t - <t>)^2 f^2(t)\, dt \bigg/ \int_{-\infty}^{+\infty} f^2(t)\, dt\,, \quad (5)$$

$$<t> = \int_{-\infty}^{+\infty} t\, f^2(t)\, dt \bigg/ \int_{-\infty}^{+\infty} f^2(t)\, dt\,, \quad (6)$$

$$\Delta\omega = \sqrt{Var(\omega)}\,, \quad (7)$$

$$Var(\omega) = \int_{-\infty}^{+\infty} (\omega - <\omega>)^2 F^2(\omega)\, d\omega \bigg/ \int_{-\infty}^{+\infty} F^2(\omega)\, d\omega\,, \quad (8)$$

$$<\omega> = \int_{-\infty}^{+\infty} \omega\, F^2(\omega)\, d\omega \bigg/ \int_{-\infty}^{+\infty} F^2(\omega)\, d\omega\,. \quad (9)$$

In Kharkevich "function moments" approach [5] the lower limit of integrals is set to zero when evaluating $\Delta\omega$.

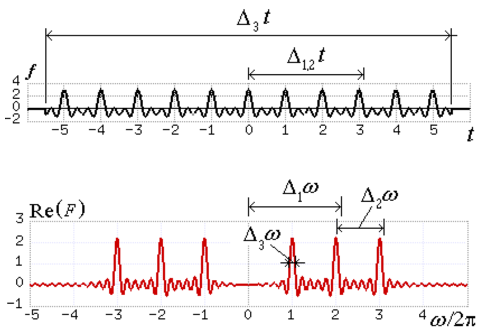

Fig.1. The "train of pulses" signal (above) and its Fourier spectrum (below). This example signal consists of three equal cosine-harmonics. Period T=1 (arbitrary unit). Due to symmetry in time, only real part of complex Fourier spectrum appears. Three deviation definitions are illustrated: 1) via conventional standard deviations; 2) Kharkevich approach that takes into account only positive frequencies; 3) pulse train duration and spectrum peak width based approach (FWHM – full width at half maximum)

It is well-known fact [1,2] that the given formulae yield the main Fourier Uncertainty Principle result:

$$\Delta t \Delta\omega \geq 0.5\,, \quad (10)$$

where the minimal number 0.5 corresponds to the gaussian function $f(t)=A\exp(-at^2)$. However, in the case of any non-gaussian signal (e.g. sine- or cosine-function), as it will be shown below by example calculations, if spectrum is not concentrated near zero frequency, the product $\Delta t\Delta\omega$ becomes remarkably greater than the lowest limit 0.5.

Kharkevich has got for his positive half-spectrum case the following result (using Bunyakovskiy inequality) [5]:

$$\Delta t\ \Delta\omega \geq 0.5/\sqrt{3} \approx 0.289 \quad (11)$$

with nearly the best value for gaussian signal $\Delta t\Delta\omega \approx 0.301$.

Below we will check given inequalities by numerical calculation and offer a new formulation for train of pulses case

$$\Delta t\ \Delta\omega \to 7.582 \quad \text{if} \quad N_{PULSE} \to \infty\,. \quad (12)$$

An example calculation

Fig. 2 compares the three calculated uncertainty products versus even number of pulses N_{PULSE} for the train of cosine-pulses. Thus only first harmonic is present, period equals T=1. The time spreading Δt is growing linearly with pulse number for all three uncertainty definition methods. The conventional Fourier-Heisenberg type uncertainty grows linearly with pulse number and is always remarkably over the theoretical minimum 0.5 as signal spectrum is far for gaussian and $\Delta\omega$ close to 1st harmonic peak $\omega_1\approx 2\pi$.

The approach of Kharkevich what uses only positive frequencies for $\Delta\omega$ calculation does not grow so rapidly, as the spectrum spreading $\Delta\omega$ decreases somewhat with increased number of pulses. However, the saturation of $\Delta t\Delta\omega$ is not observed and for practical signal processing the use of this relation remains questionable similarly to FUP.

In contrast, the 3rd uncertainty definition method approaches a limit value near 7.6. This is caused by fact that the decrease of spectrum 1st harmonic peak width due to increasing number of pulses $\Delta\omega \sim 1/N_{PULSE}$ compensates the increase of signal duration $\Delta t \sim N_{PULSE}$. This behavior seems promising to construct a useful uncertainty relation.

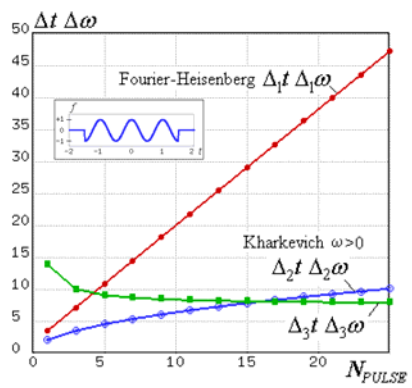

Fig. 2. The calculated uncertainty product versus number of cosine-pulses N_{PULSE}=1,3,5,... . Three uncertainty definitions are compared (see Fig.1 for explanation). Inset shows signal for N_{PULSE}=3

The train of pulses case and Mason-Zimmermann's formula

Train of pulses case is rather common for measurement applications and spectral analysis. Here it is useful for uncertainty relation development as with increasing number of pulses the decreasing of spectrum peak widths may be observed for every harmonic of signal. It is important to emphasize spectrum width for every harmonic separately as quite useless $\Delta\omega$ for FUP uses approximately distance from first harmonic peak $\omega_1\approx 2\pi$ to zero and Kharkevich definition uses spreading of full spectrum band (all harmonics) in the positive frequencies domain (Fig. 1).

The train of pulses case was earlier carefully studied, for example, by Mason and Zimmerman [8]. The analyzed the very classical problem of signal processing how the wide spectrum of the single pulse approaches the delta-function like spectrum of the periodic signal. For mathematical simplicity, they looked the even number of pulses N=1,3,5,... . As the signal period was denoted by T, they considered the limited signal in time window [NT/2, +NT/2]. So the N-pulse spectrum was calculated as standard Fourier transform:

$$F_N(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-NT/2}^{+NT/2} \exp(-i\omega\, t)\; f(t)dt \tag{13}$$

and the main result could be expressed as [8]

$$F_N(\omega) = F_1(\omega)\left[\frac{\sin(N\omega T/2)}{\sin(\omega T/2)}\right], \tag{14}$$

where spectrum for N pulses F_N is calculated from the single-pulse spectrum F_1 by using a separated delta-function forming multiplier. This square bracket term of (14) has a maximum value N for every signal harmonic $\omega_k = kT/2\pi$, k=0,±1,±2,... and describes the similar oscillating sidebands near every harmonic peak (incl. the average value, i.e. zero-frequency harmonic). The illustrating calculation is shown in Fig.3.

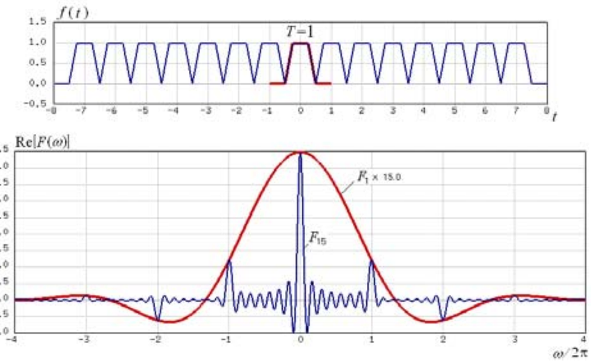

Fig. 3. Calculated train of pulses of trapezium signal (above) and its spectrum (above). The spectra of a single pulse and 15 pulses are compared (with multiplier 15 for F_1 to obtain the comparable scale). The results show how the square bracket multiplier from eq.(14) forms N-pulse spectrum from a single pulse spectrum

Definition of the applicable uncertainty relation

By performing multiple numerical calculations of Fourier transform and also by analyzing Mason-Zimmermann's formula (14) and especially its square bracket term we concluded that a reasonable uncertainty relation formulation could use signal duration and spectrum peak width for any harmonic. Preferably it could be the dominating first harmonic but similarly behaves also any harmonic which has a noticeable value in single pulse spectrum (Fig. 3). The reasonable peak width definition $\Delta\omega$ could be the FWHM (Full Width at Half Maximum) that is commonly used in different areas of spectral analysis. For the signal spreading in time Δt is reasonable to use the pulse train duration NT that serves also as the time window in measurements.

Thus, the numerical simulations supported by the theoretical considerations allowed us to come to the following uncertainty limits:

$$t_{SIGNAL} \times \Delta\omega_{FWHM} \rightarrow 7.582\,, \tag{15}$$

or

$$t_{SIGNAL} \times \Delta f_{FWHM} \rightarrow 1.2067\,, \tag{16}$$

if number of pulses N increases.

It should be emphasized that the number of pulses N and period T are not needed in those uncertainty limits if enough great number N (i.e. narrow peak widths) are reached. Theoretically, the considered $\Delta\omega$ must become enough narrow that the single pulse spectrum F_1 in (14) could be handled as a constant near respective harmonic peak. For practical estimations a satisfactory accuracy may be achieved already by rather small pulse number N=5...10, see, e.g., Fig. 2.

Discussion

Philosophically, the offered uncertainty relation (15, 16) is the mathematical result, based on the mathematical definition of the Fourier transform. This relation corresponds to the signal that is strictly periodic within a certain time window and zero outside. The uncertainty relation (15, 16) should give estimation to the achievable spectrum peak widths if the measurement apparatus obeys the mathematical Fourier transform rules. However, the peaks with final width occur in spectra not only due to limited time window but also due to unstable period, unstable amplitude, noise etc. The present relation (15, 16) corresponds to the case if we cut off from an ideal periodic signal a fragment with enough number of periods. And then spectrum peaks with final widths for all noticeable harmonics are created by the basic Fourier mathematics due to this limitation of time window.

On the other hand, as a counterexample, if any measurement apparatus would have a priori external information about the exact period of signal T then extrapolation of signal should be possible in wide time scale and eq. (15, 16) type restrictions must obtain another meaning.

Conclusion

Present work has discussed the $\Delta t \Delta\omega$ type uncertainty relations in signal processing. It has been concluded that the Fourier Uncertainty Principle that is equivalent to Heisenberg's Uncertainty Relation in quantum mechanics has not found wide application in signal processing since the evaluation of variances in frequency domain must take into account both positive and negative values of frequency.

The modification, considering only positive frequencies, as offered in 1960-ies by A. A. Kharkevich, uses as well wide spreading of signal over all harmonic peaks in frequency domain. Therefore this approach has also not found any remarkable practical application.

However, for a case of train of pulses, it is possible to define a remarkably more universal uncertainty relation $\Delta t \Delta f \rightarrow 1.2067$ that relays on signal duration and the spectrum peak width at half maximum for every influencing harmonic of signal.

In the end of our studies we found a paper [9] which refer to Kharkevich works and uncertainty problems from the viewpoint of sophisticated optical spectral analysis but not from classical signal analysis viewpoint as here. However, we kindly encourage readers notify us if eq. (5) type relations have appeared in some form in earlier studies.

Acknowledgement

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A. Udal, V. Kukk, E. Velmre. Critical Analysis of Uncertainty Relations Based on Signal Duration and Spectrum Width // Electronics and Electrical Engineering. – Kaunas: Technologija, 2009. – No. 6(94). – P. 31–34.

An essential problem of signal processing – uncertainty relations based on signal duration and spectrum width are discussed. An explanation is offered why the signal processing community has nearly forgotten the Fourier Uncertainty Principle that is totally analogous to famous coordinate-momentum Heisenberg's Uncertainty Principle. The noticeable achievements by A.A. Kharkevich from 1960-ies are critically analysed. Paper reaches to conclusion that one practically usable uncertainty relation may be defined for train of pulses type tasks. For this kind of tasks an universal uncertainty relation connecting signal duration and spectrum peak width for every harmonic is formulated. Ill. 3, bibl. 9 (in English; summaries in English, Russian and Lithuanian)..

А. Удал, В. Кукк, Э. Велмре. Критический анализ неопределенности связи с учетом длительности сигнала и диапазона спектра // Электроника и электротехника. – Каунас: Технология, 2009. – № 6(94). – С. 31–34.

Приведен анализ обработки сигналов в зависимости от длительности сигнала и диапазона спектра. Установлено, что принцип неопределенности Фурье является анализом неопределенности Гейзенберга. Исследованы основные достижения. А. А. Харкевича в данной области с 1960 года. Доказано, что связь практически возможно определить из ряда иммпульсов. Ил. 3, библ. 9 (на английском языке; рефераты на английском, русском и литовском яз.).

A. Udal, V. Kukk, E. Velmre. Signalo trukmės ir spektro pločio netikrumo ryšių kritinė analizė // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2009. – Nr. 6(94). – P. 31–34.

Išanalizuota problema, susijusi su signalų apdorojimu – signalo trukmės ir spektro pločio netikrumo ryšiais. Nustatyta, kad Furjė neapibrėžties principas yra visiškai analogiškas Heizenbergo koordinačių ir momentų neapibrėžties principui. Išnagrinėti svarbiausiai A. Charkevičiaus pasiekimai nuo 1960 m. Įrodyta, kad praktiškai taikomas ryšys gali būti nustatytas iš impulsų sekos. Il. 3, bibl. 9 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).

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