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Statistical Method of Signal – Noise Ratio Maximization

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Introduction

In the research [1] is offered an improved modification of the statistical method [2] for noisy signal detection. We shall concentrate on constructing the statistical method that provide given amplitude signal to noise maximum ratio possible, and on applying of such a method for the wider amplitude range.

According to the statistical method [2], discrete stroboscopic transformation of noisy signal proceeds as follows. Let us suppose that instantaneous amplitude of signal concealed by normally distributed noise, in the time point t_i is equal to u_i. Because of the shot noise of the input cascade of the stroboscopic converter (SC), the value

$$U_{1i} = u_{1i} + \mathrm{X}_1, \quad (1)$$

where X_1 – is normally distributed random variable with a mean

$$E\mathrm{X}_1 = 0 \quad (2)$$

and variance

$$D_1 X_1 = \sigma_1^2 \quad (3)$$

is substantively observed.

In the time point t_i noisy instantaneous signal value is n times ($n > 2$)compared with an acquainted threshold e_i. If, out of n times of comparison, U_i exceeds the threshold e_i n^+ times, then an estimation of the signal's momentary value is calculated

$$\hat{U}_i = \sigma_1 \sqrt{\frac{n-2}{n}} \Phi^{-1}(\hat{P}_n) + e_i, \quad (4)$$

where Φ – is function of standard normal distribution, Φ^{-1} – its inverse function, $\hat{P}_n = \frac{n^+ + \varepsilon(n^+)}{n}$ – estimation of threshold exceedance probability, n^+ – quantity of threshold exceedances, $\varepsilon(0) = 0.1$, $\varepsilon(n) = -0.1$ and $\varepsilon(n^+) = 0$ – in other cases.

The common factor $\sqrt{\frac{n-2}{n}}$ has sense only in the signal registration mode, and it can be omitted for signal detection.

After calculation of $\hat{U}_i$, the following value of threshold, equal to

$$e_{i+1} = e_i + \hat{U}_i \quad (5)$$

is set. The signal and threshold are compared again, and instantaneous value of signal in the next phase of the signal is calculated analogically.

The value of the standard deviation σ_1 is stored parametric variable and that's why it is considered as an acquainted and constant value. If there is no input signal, then the result of such transformation of this noise will be stochastic process with variance $D_2 = \sigma_2^2$ that depends on quantity n of signal and threshold comparison operations. The more is quantity n of comparison operations, the smaller is variance D_2. In case of transformation of centric weak signals, there can be set a constant and equal to zero threshold. In that case the result of observation of the instantaneous value can be expressed by the formula

$$\hat{U}_i = \sigma_1 \Phi^{-1}(\hat{P}_n). \quad (6)$$

Transformation of centric and concealed by noise signals takes place in case of receiving weak ultrabroadband radiolocation signals. We shall use harmonic single-oscillation with amplitude of A_1 as a signal model.

To get quite good signal-noise ratio in the output $h_2 = \frac{\bar{A}_2}{\sigma_2}$, where $\bar{A}_2$ is the amplitude of the signal in the output without zero offset, it's necessarily to have a certain quantity n of signal and threshold comparison operations (strobing). The weaker is input signal, the greater quantity of strobing is necessary. To increase the operation speed, it's reasonable to divide the operational mode of radio locator into two parts: the mode of signal detection and the mode of precise registration of signal. In the detection mode signal amplitude – noise ratio is important. In the same time one doesn't have to care about the quality of the transformed signal's form and is able to economize quantity of strobes. For that purpose there was offered the method [1] that provides higher ratio of $h_2 = \frac{\bar{A}_2}{\sigma_2}$. The essence of this method is as follows. Instead of $\hat{U}_i$ calculation according to (6) $U_i^`$ is calculated using the formula

$$U_i^` = \sigma_1 (\frac{n_i^+}{n - n_i^+ + \beta} - 1) . \quad (7)$$

Summand $\beta > 0$ is worked in, for, if there's $n_i^+ = n$, the value $U_i^`$ would not turn into infinity. In the research [1], the value β is set equal to 0.1. The signal's form in this transformation is very perturbed. But this doesn't matter for the solution of the problem.

The synthesis of method for achieving maximal signal-noise ratio.

The question may be, what kind of statistics analyzing n^+ there should be, to get the maximum possible ratio h_2 if A_1^* and n are given. It's clear, that the method synthesized in such a way will be optimal only if amplitude is A_1^*. That's why it's necessary to test quality of signal detection at wider amplitude range.

For noisy signal optimal transformation synthesis with an amplitude of A_1^* it's necessary to find such $\varepsilon(n^+)$, $0 \le n^+ \le n$, using which $\bar{A}_2 / \sigma_2$ will be maximal.

Necessitate that average value of input noise transformation is equal to 0 (i.e. the result of centric noise transformation will also be centric). Describe strobing n times in the phase of the input signal $U_1 = A_1^*$, generally, we'd like to state that as the result of transformation the average value will be equal to

$$\bar{A}_2(A_1^*) = \sigma_1 \sum_{i=0}^{n} \pi_{n,i}(A_1^*)\theta_i , \quad (8)$$

where

$$\pi_{n,i}(A_1^*) = C_n^i \Phi^i(A_1^*)(1-\Phi(A_1^*))^{n-i} \quad (9)$$

$$\theta_{n^+} = \Phi^{-1}(\frac{n^+ + \varepsilon(n^+)}{n}), \quad (10)$$

where $-n^+ < \varepsilon(n^+) < n - n^+$. These ranges allow to assume θ_{n^+} as independent real values.

In its turn transformed noise standard quadratic deviation in that case will be:

$$\sigma_2 = \sigma_1 \sqrt{\sum_{i=0}^{n} \pi_{n,i}(0)\theta_i^2} . \quad (11)$$

For the synthesis of the method, the following ratio is of interest:

$$\frac{\bar{A}_2(A_1^*) - \bar{A}_2(0)}{\sigma_2} . \quad (12)$$

It's clear, that under such θ_i, which are extreme points of (12), we will have, $\bar{A}_2(0) = 0$ and signal-noise ratio will be maximal.

It's well known, that to find extreme points of differentiable function, is enough to find corresponding derivatives of this function. After partial differentiation of (12) for each θ_i, $0 \le i \le n$, we equate the obtained derivatives to zero and will solve the system of equations relating to θ_i, $0 \le i \le n$, and will get the values θ_i, under which maximal signal-noise ratio is obtained.

Let's look upon the following example: $A_1^* = 0.75\sigma_1$ and $n = 5$ (the argumentation, why specifically such A_1^* was chosen, is stated below). In the given example it's necessary to solve the following system of equities:

$$\begin{cases} \frac{-0.03065219213}{Z} = 0.03125000000 \frac{Y\theta_0}{Z^3}, \\ \frac{-0.1460498135}{Z} = 0.1562500000 \frac{Y\theta_1}{Z^3}, \\ \frac{-0.2428831143}{Z} = 0.3125000000 \frac{Y\theta_2}{Z^3}, \\ \frac{-0.0749302863}{Z} = 0.3125000000 \frac{Y\theta_3}{Z^3}, \\ \frac{0.2491068921}{Z} = 0.1562500000 \frac{Y\theta_4}{Z^3}, \\ \frac{0.2454085141}{Z} = 0.03125000000 \frac{Y\theta_5}{Z^3}, \end{cases} \quad (13)$$

where

$$\begin{cases} Z = \sqrt{H} \\ H = 0.031250\theta_0^2 + 0.156250\theta_1^2 + 0.31250\theta_2^2 + \\ \quad + 0.31250\theta_3^2 + 0.156250\theta_4^2 + 0.031250\theta_5^2 \\ Y = -0.030652192130\theta_0 - 0.14604981350\theta_1 - \\ \quad - 0.24288311430\theta_2 - 0.07493028630\theta_3 + \\ \quad + 0.24910689210\theta_4 + 0.24540851410\theta_5 \end{cases} \quad (14)$$

As a result of solution we get we get the following values of θ_i which can be seen at column 2 of Table 1.

Tested examples allow to point out, that in such a way we get the values θ_i as a freely drawn value, each from $\theta(i) \ne 0$, multiplied by given coefficients. In the example discussed above $\theta_0 = 1$ was chosen.

Values θ_i were found analogically, in the range of $5 \le n \le 16$ (see Table 1).

Table 1. Coefficients for proposed method

i \ n	5	6	7
0	1	1	1
1	0.952948571	0.978898724	0.990481267
2	0.792384157	0.906889988	0.957998306
3	0.244453271	0.66115801	0.847149226
4	-1.625377337	-0.177409744	0.468873408
5	-8.006230454	-3.039047392	-0.82200418
6		-12.80447183	-5.227162725
7			-20.25989859

Table 1. (continuation)

i \ n	8	9	10
0	1	1	1
1	0.995694877	0.998050585	0.999116808
2	0.981003514	0.991398146	0.996102891
3	0.930868798	0.968696499	0.985817806
4	0.759782572	0.891226447	0.950719646
5	0.175945687	0.626857566	0.830946122
6	-1.816415219	-0.275309182	0.422215373
7	-8.615406659	-3.353980308	-0.972590586
8	-31.81716918	-13.86003801	-5.732407821
9		-49.71227548	-21.97542711
10			-77.40521834

Table 1. (continuation)

i \ n	11	12	13
0	1	1	1
1	0.999599769	0.99981861	0.999917787
2	0.99823397	0.999199611	0.999637235
3	0.993573141	0.99708726	0.998679841
4	0.977667918	0.989878796	0.995412705
5	0.923390868	0.9652797	0.984263507
6	0.738168806	0.881334543	0.946216534
7	0.106092956	0.594869173	0.816380095
8	-2.050884869	-0.382702504	0.373309346
9	-9.411636492	-3.718695221	-1.138682894
10	-34.53042251	-15.10287058	-6.298401823
11	-120.2490344	-53.95171095	-23.90609767
12		-186.5245456	-83.99288865
13			-289.040868

Table 1. (continuation)

i \ n	14	15	16
0	1	1	1
1	0.999962738	0.999983111	0.999992345
2	0.999835578	0.999925476	0.999966222
3	0.999401644	0.999728794	0.999877075
4	0.997920827	0.999057613	0.99957286
5	0.992867494	0.996767185	0.998534717
6	0.975622846	0.988951033	0.994992024
7	0.916774965	0.96227818	0.982902473
8	0.715954805	0.871256266	0.941646521
9	0.030649976	0.560641252	0.800859343
10	-2.307973333	-0.499341815	0.320418869
11	-10.28859563	-4.116565873	-1.319098678
12	-37.52270895	-16.46045162	-6.914001556
13	-130.4599387	-58.58433256	-26.00677569
14	-447.611027	-202.3333415	-91.16144091
15		-692.8811524	-313.5036932
16			-1072.253308

Corresponding theoretically calculated signal-noise ratio under $A_1^* = 0.75\sigma_1$ for amplitude of input signal $A_1 = 0.50\sigma_1$, $A_1 = 0.75\sigma_1$, $A_1 = 1.00\sigma_1$, $A_1 = 1.25\sigma_1$ and $A_1 = 1.50\sigma_1$ are stated in the Table 2.

Table 2. Signal – noise ratios for proposed method

n	0.50	0.75	1.00	1.25	1.50
5	0.97	1.64	2.36	3.05	3.62
6	1.09	1.95	2.93	3.89	4.74
7	1.22	2.29	3.59	4.94	6.16
8	1.34	2.67	4.37	6.23	7.97
9	1.47	3.09	5.30	7.84	10.30
10	1.56	3.56	6.42	9.85	13.29
11	1.73	4.09	7.76	12.36	17.15
12	1.87	4.70	9.36	15.51	22.12
13	2.01	5.38	11.29	19.46	28.55
14	2.16	6.16	13.61	24.41	36.84
15	2.32	7.04	16.41	30.62	47.56
16	2.48	8.04	19.77	38.42	61.42

For reference, there are theoretically calculated signal-noise ratios according to the method [1], but in Table 4 – according the method [2].

Table 3. Signal – noise ratios for n^+/n^- method

n	0.5	0.75	1	1.25	1.5
5	0.81	1.51	2.33	3.18	3.93
6	0.88	1.76	2.89	4.11	5.26
7	0.95	2.03	3.54	5.29	7.00
8	1.03	2.33	4.31	6.76	9.26
9	1.11	2.68	5.24	8.60	12.20
10	1.22	3.07	6.34	10.89	16.00
11	1.34	3.52	7.62	13.71	20.84
12	1.48	4.01	9.11	17.10	26.90
13	1.65	4.56	10.76	21.07	34.27
14	1.84	5.12	12.54	25.53	42.93
15	2.04	5.68	14.32	30.28	52.59
16	2.24	6.19	15.98	35.00	62.74

Table 4. Signal – noise ratios for method [2]

n	0.5	0.75	1	1.25	1.5
5	0.87	1.30	1.70	2.04	2.31
6	0.97	1.46	1.92	2.34	2.68
7	1.05	1.60	2.13	2.62	3.03
8	1.13	1.72	2.32	2.87	3.35
9	1.21	1.84	2.49	3.11	3.65
10	1.28	1.95	2.64	3.32	3.92
11	1.34	2.05	2.78	3.51	4.17
12	1.40	2.14	2.91	3.68	4.40
13	1.45	2.22	3.02	3.84	4.61
14	1.51	2.30	3.13	3.99	4.81
15	1.56	2.37	3.24	4.13	4.99
16	1.61	2.45	3.33	4.26	5.16

From the obtained results we can see, that under chosen amplitude of $A_1^* = 0.75\sigma_1$ and amplitudes of input signal close to A_1^* the synthesized method provides higher signal-noise ratio. For example, under $A_1^* = 0.75\sigma_1$ and $n = 16$ the synthesized method provides signal-noise ratio equal to 8.04, the method [1] in the same circumstances has $h_2 = 6.19$, but the method [2] – $h_2 = 2.45$. So an advantage, regarding the signal detection, in one case is 23%, and in the other – 70%. This advantage won't be so seen on the individual involutes, but it's important for the statistical mean. The method [1] provides insignificantly better signal-noise ratio, comparing to the given method, just for moderately high amplitudes of input signal. Although this advantage doesn't have really important practical meaning, for high amplitude signal detection is not technically complicated.

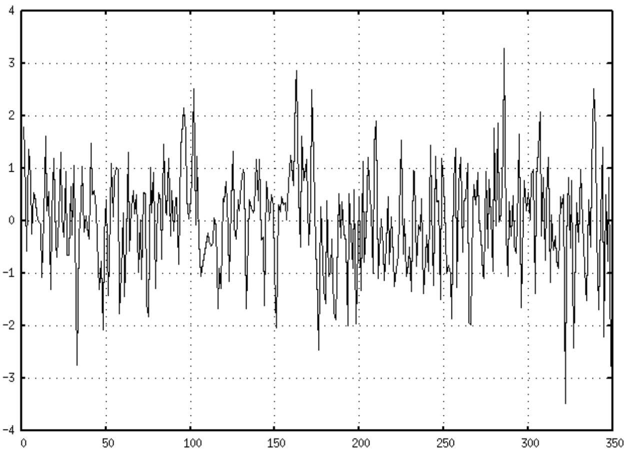

Fig. 1a. Input signal with amplitude of $A_1 = 0.75\sigma_1$

In general case it can be any positive number, taken as A_1^*. Although in this case obtained coefficients θ_i will be optimal specifically for this signal amplitude. On this basis A_1^* was chosen from relatively small $A_1^* = 0.75\sigma_1$, for weak signals detection is especially difficult. As an example on the Fig.1 can be seen input signal concealed by noise with amplitude of $A_1 = 0.75\sigma_1$.

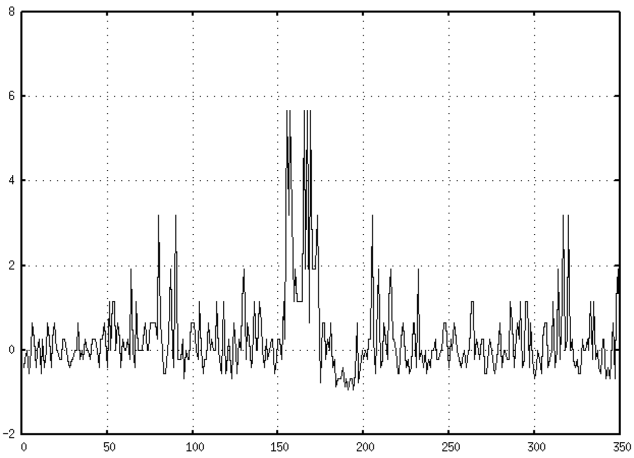

Fig. 1b. The signal transformed by the method [1]

As it's seen on the figure, single-oscillation of such amplitude is completely concealed by noise. On the figure 1b is stated the signal, transformed according to the method [1], but on the figure 1c - the signal, transformed according to the method [2]. The other conditions of simulation were as follows: quantity of strobing phase points for the period of single-oscillation $n_T = 50$, involute length $N_T = 350$, input signal location – in the center of involute (phase points from 150 to 200).

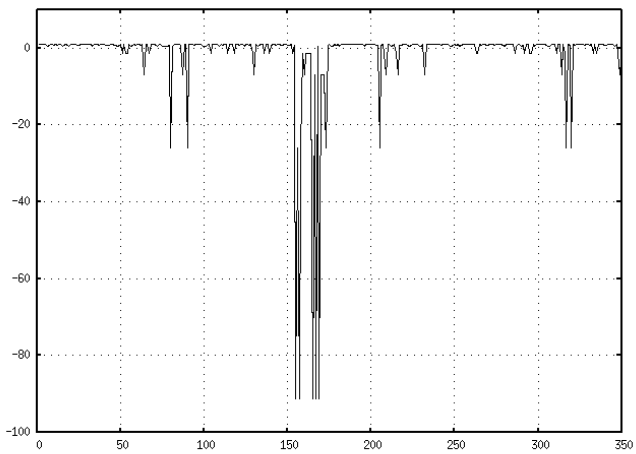

Fig. 1c. The same signal, transformed by the given method

The synthesized method provides maximum possible signal-noise ratio for each amplitude $A_1 = A_1^*$. That's why the synthesized method can be used for quality estimation of any other signal detection method by the criterion A_2/σ_2.

In the Table 1 there are also coefficients θ_i for very small quantity of signal and threshold comparison operations. The results of such n don't let to detect small amplitude signals without additional processing of transformed signal. As an additional processing one can use a convolution with an according standard.

To obtain the best results after correlation filter, we will find the analytic expression of the single-oscillation transformation result with a certain A_1.

It should be noted that this standard will be the best only under this detected signal amplitude. That's why we will choose the amplitude A_1 that is situated on the boundary of signal detection and omission.

Then under the amplitudes nearby this threshold amplitude we will obtain the improved signal – noise ratio correlation. Under the greater input signal amplitudes this standard optimality is not essential, since these signals can be detected without any efforts.

We will find the form of the standard under certain given A_1, $\sigma_1 = 1$, n and n_T. And we will use formulas (8), (9) and (10) for this purpose, according to them the mean observation of the transformation result that is obtained using statistical method is expressed as follows:

$$b_i(n, u_i) = \sum_{j=0}^{n} \pi_n(j)\theta_j \,. \qquad (15)$$

The single-oscillation transformation result that is obtained using the method recommended is not symmetrical. That's why it's necessary to form the mirror image of the functional relation $b_i(n,u_i)$ to obtain the $b_i^*(n,u_i)$ standard:

$$b_i^*(n,u_i) = -b_{n_T-i+1}(n,u_{n_T-i+1}) . \qquad (16)$$

The form of the standard for the method recommended obtained under the following scenarios: $A_1 = 0.75$, $\sigma_1 = 1$, $n = 5$ and $n_T = 50$, is shown in the Fig. 2. As we can see, the standard obtained differs greatly from the form of the harmonical single-oscillation.

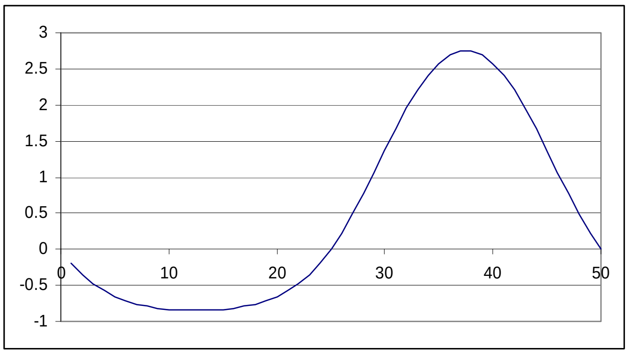

Fig. 2. The standard functional relation under $A_1 = 0.75$, $\sigma_1 = 1$, $n = 5$ and $n_T = 50$

On the Fig. 3 there can be seen the result of correlation filtration of the signal with an amplitude of $A_1 = 0.75\sigma_1$ using $n = 5$ after convolution with a half-wave of harmonic single-oscillation. In the center of the involute the detected signal is clearly seen.

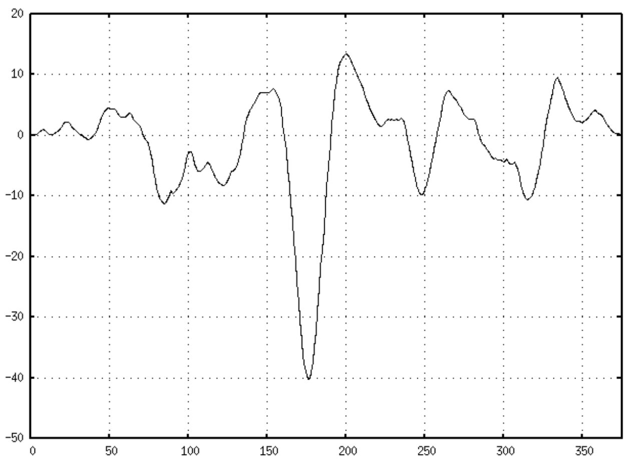

Fig. 3. The result of the convolution of the transformed signal with according standard using $A_1 = 0.75\sigma_1$, $n = 5$, $n_T = 50$

Since the input signal amplitude is indeterminate, we shall check what the $\bar{A}_3/\sigma_3$ ratio will be in case of nonoptimal standard. We will use the following standards: under $A_1 = 0.25$, $A_1 = 0.50$, $A_1 = 0.75$ and $A_1 = 1.00$, and we will change the amplitude of the input signal from $A_1 = 0.25$ to $A_1 = 1.50$

Table 5. $\bar{A}_3/\sigma_3$ ratios with n=5 signal-threshold comparison operations for proposed method

	0.25	0.5	0.75	1.0
0.25	1.7044	1.6848	1.6464	1.6094
0.5	3.5938	3.6356	3.6136	3.5742
0.75	5.666	5.8303	5.8657	5.8496
1	7.8283	8.1504	8.2675	8.2903
1.25	9.9329	10.422	10.628	10.697
1.5	11.836	12.48	12.77	12.884

Table 6. $\bar{A}_3/\sigma_3$ ratios with n=16 signal-threshold comparison operations for proposed method

	0.25	0.5	0.75	1.0
0.25	1.6811	1.646	1.6166	1.604
0.5	7.0909	7.2424	7.2206	7.2018
0.75	21.654	22.45	22.518	22.508
1	52.282	54.489	54.773	54.796
1.25	102.76	107.17	107.74	107.79
1.5	169.44	176.39	177.09	177.1

$\bar{A}_3/\sigma_3$ ratios for the n^+/n^- method using the same simulation scenarios are listed in the Tables 7 and 8 for $n = 5$ and $n = 16$ respectively.

Table 7. $\bar{A}_3/\sigma_3$ ratios with $n = 5$ signal-threshold comparison operations for the n^+/n^- method

	0.25	0.5	0.75	1.0
0.25	1.2063	1.1844	1.1502	1.1242
0.5	2.7747	2.8259	2.8071	2.7799
0.75	4.7902	4.9902	5.0236	5.0127
1	7.1699	7.5682	7.6769	7.6935
1.25	9.6981	10.313	10.506	10.555
1.5	12.124	12.945	13.216	13.297

Table 8. $\bar{A}_3/\sigma_3$ ratios with $n = 16$ signal-threshold comparison operations for the n^+/n^- method

	0.25	0.5	0.75	1.0
0.25	2.891	2.8196	2.6423	2.4793
0.5	7.3096	7.4946	7.348	7.1054
0.75	16.555	17.758	18.113	17.968
1	37.534	41.493	43.415	43.766
1.25	78.456	87.995	93.151	94.569
1.5	141.81	159.85	169.74	172.59

As you can see in the tables, the single-oscillation amplitude chosen used to obtain the standard is not critical for the relatively broad input signals amplitudes range. Apparently, you can see that the method recommended almost always provides higher h_3 , comparing to the n^+/n^- method.

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The method for discrete stroboscopic ultrabroadband radar-location noisy signal detection and transformation is recommended and explained. The advantages of this method, comparing to the n^+/n^- method, are illustrated from the point of view of the higher noise – signal ratio before additional processing of the stroboscopically transformed signals and after that. Ill. 3, bibl. 3 (in English; summaries in English, Russian and Lithuanian).

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Предлагается и исследуется метод дискретного стробоскопического преобразования и обнаружения зашумленных сигналов сверхширокополосной радиолокации. Иллюстрируется преимущество предлагаемого метода по сравнению с методом n^+/n^- с точки зрения обеспечения более высокого отношения сигнала к шуму как до, так и после дополнительной обработки стробоскопически преобразованных сигналов. Ил. 3, библ. 3 (на английском языке; рефераты на английском, русском и литовском яз.).

V. Plociņš. Statistinis signalo ir triukšmo santykio maksimizavimo metodo tyrimas // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2009. – Nr. 6(94). – P. 3–8.

Siūlomas ir analizuojamas diskretinių stroboskopinių ultraplačiajuosčių radaro lokacinės sistemos signalų detektavimo, esant triukšmams, metodas. Metodo pranašumai, palyginti su n^+/n^- metodu, pateikti atsižvelgiant į galimybę užtikrinti didesnį signalo ir triukšmo santykį prieš papildomai apdorojant stroboskopiškai transformuotus signalus ir juos apdorojus. Il. 3, bibl. 3 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).