http://dx.doi.org/10.5755/j01.eee.19.5.4368 ELEKTRONIKA IR ELEKTROTECHNIKA, ISSN 1392-1215, VOL. 19, NO. 5, 2013

Constructed Polynomial Windows with High Attenuation of Sidelobes

K. Okarma[1]
[1]*Department of Signal Processing and Multimedia Engineering, Faculty of Electrical Engineering, West-Pomeranian University of Technology, Szczecin, Sikorskiego 37, 70-313 Szczecin, Poland*
okarma@zut.edu.pl

***Abstract*—In the paper the idea of the constructed polynomial windows is presented together with the obtained results of their optimization towards low level of the sidelobes. The main advantages of the polynomial windows are their low computational complexity and ability to change their frequency properties by some modifications of the polynomial coefficients in the time domain. Both of these merits are preserved in the constructed polynomial windows family discussed in the paper and the additional profit is the possibility of using two lower order polynomials with additional switching operation with similar sidelobes' attenuation as for higher order standard polynomial windows.**

***Index Terms*—Polynomial windows, window functions, spectrum analysis.**

I. INTRODUCTION

Typical applications of the window functions in digital signal processing are related to spectral analysis based on Fourier transform and design of Finite Impulse Response (FIR) filters for the reduction of Gibbs phenomenon. Considering the Fourier analysis, the main advantage of using windows is the reduction of the spectral leakage caused by the finite number of processed samples in the time domain, due to the prevention of the rapid cutting of the series of collected data samples.

Window functions can be characterised by several parameters, which influence their characteristics in frequency domain. A perfect window in the frequency domain should have a narrow mainlobe, low level of sidelobes and high sidelobes decay ratio. Nevertheless, these requirements are contradictory so many window functions partially fulfilling them have been proposed by various researchers.

Classical well-known windows, used instead of the rectangular or simple Bartlett window, are Hann, Hamming and Blackman windows [1] and their modifications [2] based on raised-cosine functions. Some other famous windows are Dolph-Chebyshev and Kaiser, which have been optimized using various criteria - the first one is the effect of the minimization of the sidelobes level without decay and the second one is sub-optimal in the energetic sense minimizing the sidelobes to the mainlobe energy ratio.

Manuscript received January 07, 2013; accepted April 02, 2013.

In recent years some new windows have been designed in order to meet some particular requirements. Some of them can be controlled using a specific parameter e.g. as a trade-off between the mainlobe and sidelobes peak. This can be done for the unispherical windows [3] based on Gegenbauer polynomials as well as for windows proposed by Zierhofer [4].

Some other recent proposed ideas are the application of cosine hyperbolic window [5] as well as the sinc function based window [6] and the modified Hamming window [7]. None of those windows is optimized towards low sidelobes level, since e.g. window presented in the paper [8] is based on the minimization of the energetic criterion, similarly as Kaiser window, and the recently proposed modification of Hamming window [7] is still based on the raised-cosine windows family.

The sidelobes level –44.7 dB reported in this paper can also be obtained using the polynomial window, assuming the same mainlobe width (8-th and 10-th order polynomial windows with better sidelobes rejection have been reported in the paper [9]). Another idea, presented recently, is the extension of the polynomial windows family into rational polynomial windows leading to good results in the energetic sense [10] with low computational complexity.

Another type of windowing functions, known as B-Spline windows, which may be considered as quite similar to the proposed family, has been discussed in the paper [11]. Unfortunately, the properties of such windows are worse than the polynomial windows e.g. –39 dB sidelobes level is obtained for the width of the main lobe (WML) equal to 3. Further decrease of the sidelobes level for B-Spline windows causes the significant increase of the mainlobe width.

Nevertheless, the use of many known windows in DSP applications, even some of the recently proposed indicated above, may be troublesome, especially for the varying number of samples, since the pre-computing of their values stored in the memory is necessary. Alternatively, the expansion of the cosine or Bessel function into series may be needed. However, the use of low order expansion may be insufficient and can change the frequency properties of the window. Using the expansion into N-th order polynomial, according to Horner's scheme, $2 \cdot N$ arithmetic operations are necessary for its calculation at each point.

In order to reduce such complexity, low order polynomial windows family has been proposed [12] and optimized towards high sidelobes attenuation [9], since an important limitation of the polynomial windows presented in the paper [12] is very high level of the first sidelobe especially for windows with narrow mainlobe. A similar class of polynomial windows for narrowband signal processing has been proposed earlier [13].

Such windows, due to low order of polynomial, have narrow mainlobe but relatively high sidelobes peak level, so their applicability for the suppression of near-by interferences is strongly limited and their main purpose can be related to narrowband filtering with distant interference rejection. Since lower sidelobes peak level requires higher polynomial order, windows proposed in the papers [9], [11] are more suitable for the suppression of close-by tones. The main motivation of this paper is the development of a new window family based on polynomial windows with easily changing properties and even lower computational cost.

II. Polynomial Window

Standard symmetrical (2·N)-th order polynomial window with only even exponents considered in the time interval from −T/2 to T/2 is defined as

$$p_{2N}(t) = 1 + \sum_{n=1}^{N} C_{2n}\left(\frac{t}{T}\right)^{2n}. \tag{1}$$

Frequency response of such window can be expressed as [10]

$$W(f) = I_0 + \sum_{n=1}^{N} C_{2n} I_{2n}, \tag{2}$$

where I_{2n} is defined in the recurrent form

$$I_0 = \frac{\sin(\pi \cdot f \cdot T)}{\pi \cdot f}, \tag{3}$$

$$I_{2n} = \frac{T\left[(\pi \cdot f \cdot T)\cdot \sin(\pi \cdot f \cdot T) + 2n \cdot \cos(\pi \cdot f \cdot T)\right]}{4^n \cdot (\pi \cdot f \cdot T)^2} - \frac{n\cdot(2n-1)}{2\cdot(\pi \cdot f \cdot T)^2}\cdot I_{2n-2}. \tag{4}$$

As noticed in the paper [9], an additional normalizing factor should be used forcing W(0) = 1 (0 dB), which can be expressed as

$$W(0) = 1 + \sum_{n=1}^{N} C_{2n} K_{2n}, \tag{5}$$

where

$$K_{2n} = \lim_{f \to 0} I_{2n}. \tag{6}$$

Multiplying both sides of the eq. (4) by $(\pi f \mathrm{T})^2$ for $f = 0$ and $T = 1$ we obtain

$$\frac{2n}{4^n} - \frac{n(2n-1)}{2}\cdot K_{2n-2} = 0, \tag{7}$$

so finally the expression (5) is calculated as

$$W(0) = 1 + \sum_{n=1}^{N} \frac{C_{2n}}{4^n(2n+1)}. \tag{8}$$

III. Proposed Constructed Polynomial Window

The standard polynomial window (1) can be extended into constructed polynomial window assuming the stick points $t_{s1} = -T/4$ and $t_{s2} = T/4$, as:

$$c_{2N,2M}(t) = \begin{cases} 1 + \sum\limits_{n=1}^{N} a_{2n}\left(\frac{t}{T}\right)^{2n}, & for \quad |t| < \frac{T}{4}, \\ b_0 + \sum\limits_{m=1}^{M} b_{2m}\left(\frac{t}{T}\right)^{2m}, & for \quad |t| \geq \frac{T}{4}, \end{cases} \tag{9}$$

where two (2·N)-th and (2·M)-th order polynomials are used in internal and external parts of the window respectively.

The control of the sidelobes decay speed of such family of windows can be performed similarly as for the standard polynomial windows forcing the window values and the consecutive derivatives to zero at $t = \pm T/2$, but these limitations are related only to the coefficients b_{2m} representing the external parts of the window. Assuming 12 dB/octave asymptotic decay of sidelobes only the window value should be forced to zero so $c(t = \pm T/2) = 0$ and each derivative set to 0 for $t = \pm T/2$ causes the increase of this ratio by 6 dB/octave. In the consequence the following system of equations is obtained, assuming the sidelobes decay ratio equal to 24 dB/octave (two derivatives set to 0) for the 10-th order window:

$$\begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} \\ 0 & 1 & \frac{1}{2} & \frac{3}{16} & \frac{1}{16} & \frac{5}{256} \\ 0 & 2 & 3 & \frac{15}{8} & \frac{7}{8} & \frac{45}{128} \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_2 \\ b_4 \\ b_6 \\ b_8 \\ b_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

so finally the following equations should be used for the calculation of some coefficients as the functions of the others:

$$\begin{cases} b_0(b_6, b_8, b_{10}) = -\frac{1}{64}\cdot b_6 - \frac{3}{256}\cdot b_8 - \frac{3}{512}\cdot b_{10}, \\ b_2(b_6, b_8, b_{10}) = \frac{3}{16}\cdot b_6 + \frac{1}{8}\cdot b_8 + \frac{15}{256}\cdot b_{10}, \\ b_4(b_6, b_8, b_{10}) = -\frac{3}{4}\cdot b_6 - \frac{3}{8}\cdot b_8 - \frac{5}{32}\cdot b_{10}. \end{cases} \tag{11}$$

Depending on the number of fulfilled conditions (10) various asymptotic sidelobes decay ratios may be achieved and similar calculations may be conducted for other decay ratios using the reduced system of equations (10). Such obtained windows are marked as $cD_{2n,2m}$ where D is the decay ratio in dB/octave.

Nevertheless for constructed windows the requirements expressed in the system (10) are not enough to guarantee the assumed decay ratios, since another constraint is related to the stick points $t_{s1} = -T/4$ and $t_{s2} = T/4$, where the inner and outer polynomials should be equal even for 6 dB/octave asymptotic sidelobes decay. Each derivative's equality at the stick points allows obtaining 6 dB/octave higher decay speed (e.g. 24 dB/octave requires equal values of three derivatives). Solving the appropriate system of equations the following limitations are obtained in similar form as for the outer polynomial:

$$\begin{cases} a_2 = 16\cdot(b_0-1)+b_2+\dfrac{b_4-a_4}{16}+\dfrac{b_6-a_6}{256}+\dfrac{b_8-a_8}{4096}+\dfrac{b_{10}-a_{10}}{65536}, \\ a_4 = 256\cdot(1-b_0)+b_4+\dfrac{b_6-a_6}{8}+\dfrac{b_8-a_8}{256}+\dfrac{b_{10}-a_{10}}{1024}, \\ a_6 = 4096\cdot(b_0-1)+b_6+\dfrac{3\cdot(b_8-a_8)}{4096}+\dfrac{3\cdot(b_{10}-a_{10})}{128}, \\ a_8 = 65536\cdot(1-b_0)+b_8+\dfrac{b_{10}-a_{10}}{4}. \end{cases} \quad (12)$$

Since some of the coefficients are calculated as the functions of the others, the remaining ones can be optimized together with the independent coefficients of the outer polynomial (not restricted by the specified number of conditions (10) for given asymptotic decay speed). The number of the fulfilled conditions for the inner polynomial (12) depends on the desired decay ratio.

Using the arbitrarily chosen values of the coefficients from the right side of the system (11), considering also the stick points limitations (12), the optimization of the window function can be done using various criteria. Minimizing the sidelobes level, the obtained solution for the outer polynomial is

$$\{b_6, b_8, b_{10}\}_{opt} = \arg\min_{b_6, b_8, b_{10}} \left\{ \max_{f>WML} \left(\left| W(f, b_6, b_8, b_{10}) \right| \right) \right\}, \quad (13)$$

where *WML* denotes the width of the mainlobe.

IV. RESULTS OF OPTIMIZATION

Performing the optimization for the 1024-point windows being a good approximation of the continuous windows (typically about 0.1 dB error can be observed in the frequency characteristics) some interesting results have been obtained.

Specifically, the use of some lower order constructed windows may lead to similar results as using higher order standard polynomial ones, so their application seems to be more efficient, since the only additional operation in comparison to the polynomial window of the same order is the switching at $t = \pm T/4$. The results are presented in Table I–Table III and the most interesting frequency domain characteristics are shown in Fig. 1–Fig. 3.

TABLE I. POLYNOMIAL COEFFICIENTS AND FREQUENCY PROPERTIES OF THE POLYNOMIAL (p) AND COMBINED (c) WINDOWS OPTIMIZED WITHOUT ANY LIMITATIONS EXCEPT THE CONTINUITY AT t = ±T/4 (6 DB/OCTAVE DECAY).

Window type	**HSLL [dB]**	**WML [1/T]**	**Polynomials' coefficients**										
			b_0	**a_2**	**b_2**	**a_4**	**b_4**	**a_6**	**b_6**	**a_8**	**b_8**	**a_{10}**	**b_{10}**
$p6_2$	-21.4	1.43	---	-4	---	---	---	---	---	---	---	---	---
$c6_{2,2}$	-30.7	1.75	0.7384	-7.2925	-3.1073	---	---	---	---	---	---	---	---
$p6_4$	-39.4	1.94	---	-8.2445	---	18.9840	---	---	---	---	---	---	---
$c6_{4,4}$	-47.8	2.19	0.8498	-10.4630	-6.8982	33.2288	14.6543	---	---	---	---	---	---
$p6_6$	-48.6	2.36	---	-11.6844	---	50.4560	---	-78.8550	---	---	---	---	---
$c6_{6,6}$	-58.7	2.78	0.8829	-15.5208	-11.3517	103.3766	50.5970	-334.2564	-77.5950	---	---	---	---
$p6_8$	-62.8	2.80	---	-14.5628	---	89.5751	---	-276.7821	---	353.9600	---	---	---
$c6_{8,8}$	-74.4	3.125	0.9583	-16.5314	-14.7362	118.8997	91.3646	-409.0902	-269.1787	122.2675	315.9845	---	---
$p6_{10}$	-71.8	3.03	---	-16.2559	---	116.4718	---	-470.4690	---	1071.089	---	-1037.302	---
$c6_{10,10}$	-75.5	3.125	0.9517	-16.7670	-14.7272	122.2326	91.4412	-418.5111	-269.1165	-74.1387	314.2194	1345.401	-37.5616

TABLE II. POLYNOMIAL COEFFICIENTS AND FREQUENCY PROPERTIES OF THE POLYNOMIAL (p) AND COMBINED (c) WINDOWS OPTIMIZED FOR 12 DB/OCTAVE DECAY RATIO.

Window type	**HSLL [dB]**	**WML [1/T]**	**Polynomials' coefficients**										
			b_0	**a_2**	**b_2**	**a_4**	**b_4**	**a_6**	**b_6**	**a_8**	**b_8**	**a_{10}**	**b_{10}**
$p12_4$	-27.7	1.83	---	-8	---	16	---	---	---	---	---	---	---
$c12_{4,4}$	-36.1	2.094	0.8736	-11.4541	-7.4091	48.0191	15.6589	---	---	---	---	---	---
$p12_6$	-48.6	2.35	---	-11.691	---	50.513	---	-79.003	---	---	---	---	---
$c12_{6,6}$	-56.9	2.656	0.9127	-14.1834	-11.1797	76.8676	47.7862	-124.5649	-70.6832	---	---	---	---
$p12_8$	-55.8	2.75	---	-15.010	---	92.183	---	-273.039	---	321.879	---	---	---
$c12_{8,8}$	-66.9	3.095	0.9504	-17.2736	-14.9269	141.5566	92.9232	-837.7307	-270.5143	-3291.819	307.3159	---	---
$p12_{10}$	-70.5	3.00	---	-16.084	---	113.9596	---	-457.8596	---	1049.839	---	-1073.507	---
$c12_{10,10}$	-71.5	3.125	0.9930	-16.5472	-16.2544	119.0767	11.6086	-441.9404	-462.7347	593.9409	1036.388	379.9337	-1027.03

TABLE III. POLYNOMIAL COEFFICIENTS AND FREQUENCY PROPERTIES OF THE POLYNOMIAL (p) AND COMBINED (c) WINDOWS OPTIMIZED FOR 18 DB/OCTAVE DECAY RATIO.

Window type	**HSLL [dB]**	**WML [1/T]**	**Polynomials' coefficients**										
			b_0	**a_2**	**b_2**	**a_4**	**b_4**	**a_6**	**b_6**	**a_8**	**b_8**	**a_{10}**	**b_{10}**
$p18_6$	-33.4	2.22	---	-12	---	48	---	-64	---	---	---	---	---
$c18_{6,6}$	-39.0	2.19	0.9334	-12.8078	-9.6126	83.2187	32.0962	-306.9758	-34.3225	---	---	---	---
$p18_8$	-55.8	2.75	---	-14.990	---	91.937	---	-271.983	---	320.293	---	---	---
$c18_{8,8}$	-64.0	3.03	0.9539	-17.0583	-15.0413	119.3588	93.4015	-306.8264	-269.4199	-506.5485	301.7082	---	---
$p18_{10}$	-62.5	3.13	---	-18.267	---	143.302	---	-605.988	---	1365.73	---	-1286.1	---
$c18_{10,10}$	-68.0	3.28	0.9920	-18.8347	-18.2890	157.4713	143.9399	-744.5911	-607.593	1690.939	1363.787	558.6703	-1296.914

An interesting property of the proposed windows is the fact that higher sidelobes attenuation does not necessarily cause a significant increase of the mainlobe width. In some cases optimized constructed windows have narrower mainlobe and lower sidelobes level than higher order polynomial windows. It can be noticed e.g. comparing the characteristics of $c12_{6,6}$ and $p12_8$ windows or $c18_{8,8}$ and $p18_{10}$ ones, which are very similar to each other (the comparison is shown in Fig. 3). The use of such constructed windows may lead to similar properties for lower order of polynomials reducing the computational complexity according to Horner's scheme by 2 arithmetic operations for each sample in comparison to standard polynomial windows.

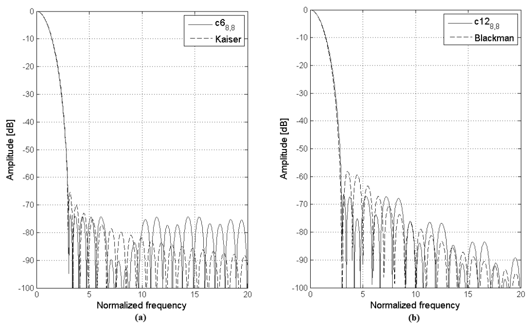

Fig. 1. Comparison of the frequency characteristics of chosen constructed polynomial windows with Kaiser and Blackman windows.

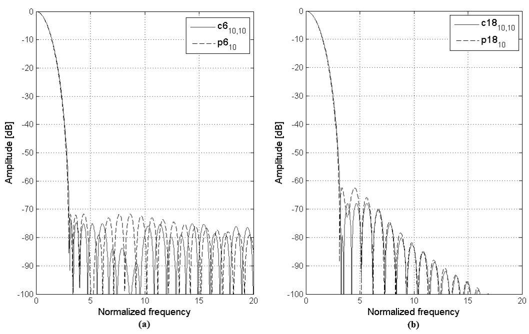

Fig. 2. Comparison of the frequency characteristics of chosen 10-th order polynomial and constructed polynomial windows.

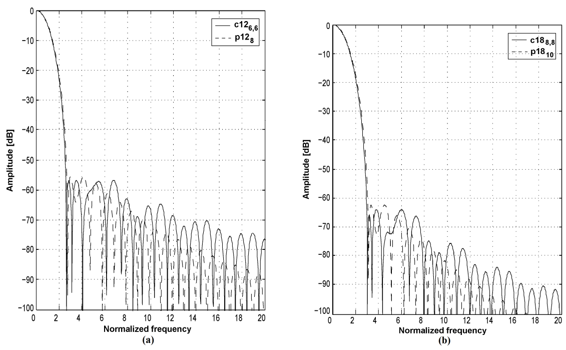

Fig. 3. Illustration of the frequency characteristics' similarity for chosen polynomial and lower order constructed windows with better properties.

V. Conclusions

Due to their low computational complexity both polynomial and constructed polynomial windows can be an interesting alternative to classical raised-cosine windows, especially for the adaptive windowing.

Polynomial functions can also be used in for some other purposes related to signal processing e.g. efficient ECG modelling [14] or some other computer science applications such as curve fitting [15].

Presented family of constructed windows allow to achieve even better sidelobes attenuation than standard polynomial windows for lower order of polynomials. Nevertheless the use of the higher order windows may be troublesome in hardware implementations due to relatively large values of the coefficients.

The proposed optimization of the constructed polynomial windows for higher decay ratios may lead to relatively wide mainlobes but their application instead of the previously proposed polynomial windows results in better properties in terms of sidelobes attenuation. The proposed family of windows can also be applied for the fixed mainlobe width e.g. corresponding to Hamming (WML = 2) or Blackman windows (WML = 3). For this purpose additional constraints should be applied during the optimization procedure, similarly as for standard polynomial windows [9].

References

[1] F. Harris, "On the use of windows for harmonic analysis with discrete Fourier transform", in *Proc. of the IEEE*, vol. 66, no. 1, pp. 51–83, Jan. 1978. [Online]. Available: http://dx.doi.org/10.1109/PROC.1978.10837

[2] A. Nuttall, "Some windows with very good sidelobe behavior", *IEEE Trans. Acoust. Speech Signal Process.*, vol. 29, no. 1, pp. 84–91, Feb. 1981. [Online]. Available: http://dx.doi.org/10.1109/TASSP.1981.1163506

[3] A. G. Deczky, "Unispherical windows," in *Proc. Int. Symp. Circuits and Systems (ISCAS 2001)*, Sydney, 2001, pp. 85–88.

[4] C. Zierhofer, "Data window with tunable side lobe ripple decay", *IEEE Signal Processing Letters*, vol. 14, no. 11, pp. 824–827, Nov. 2007. [Online]. Available: http://dx.doi.org/10.1109/LSP.2007.901696

[5] K. Avci, A. Nacaroglu, "Cosh window family and its application to FIR filter design", *AEU - Int. J Electronics and Communications*, vol. 63, no. 11, pp. 907–916, Nov. 2009. [Online]. Available: http://dx.doi.org/10.1016/j.aeue.2008.07.013

[6] M. Shayesteh, M. Mottaghi-Kashtiban, "FIR filter design using a new window function", in *Proc. 16th Int. Conf. Digital Signal Process. (DSP 2009)*, Santorini, 2009, pp. 1–6.

[7] M. Mottaghi-Kashtiban, M. Shayesteh, "New efficient window function, replacement for the Hamming window", *IET Signal Processing*, vol. 5, no. 5, pp. 499–505, Aug. 2011. [Online]. Available: http://dx.doi.org/10.1049/iet-spr.2010.0272

[8] A. Rowinska-Schwarzweller, M. Wintermantel, "On designing FIR filters using windows based on Gegenbauer polynomials", in *Proc. of the IEEE Int. Symp. Circuits and Systems (ISCAS 2002)*, Scottsdale, 2002, vol. 1, pp. 413–416.

[9] K. Okarma, "Polynomial windows with low sidelobes' level", *Signal Processing*, vol. 87, no. 4, pp. 782–788, Apr. 2007. [Online]. Available: http://dx.doi.org/10.1016/j.sigpro.2006.09.007

[10] K. Okarma, "Rational polynomial windows as an alternative for Kaiser window", *Elektronika ir Elektrotechnika (Electronics and Electrical Engineering),* no. 7, pp. 91–94, 2011.

[11] K. Toraichi, M. Kamada, S. Itahashi, R. Mori, "Window functions represented by B-spline functions", *IEEE Trans. Acoust. Speech Signal Process.*, vol. 37, no. 1, pp. 145–147, Jan. 1989. [Online]. Available: http://dx.doi.org/10.1109/29.17517

[12] R. G. Kulkarni, "Polynomial windows with fast decaying sidelobes for narrow-band signals", *Signal Processing*, vol. 83, no. 6, pp. 1145–1149, Jun. 2003. [Online]. Available: http://dx.doi.org/10.1016/S0165-1684(03)00019-7

[13] R. Kulkarni, "Polynomials for narrowband signal processing", *IEE Proc. - Vis. Image Signal Process.*, vol. 149, no. 3, pp. 159–161, Jun. 2002. [Online]. Available: http://dx.doi.org/10.1049/ip-vis:20020384

[14] S. Jokic, V. Delic, Z. Peric, S. Krco, D. Sakac, "Efficient ECG modeling using polynomial functions", *Elektronika ir Elektrotechnika (Electronics and Electrical Engineering),* no. 4, pp. 121–125, Apr. 2011.

[15] B. Cetisli, H. Kalkan, "Polynomial curve fitting with varying real powers", *Elektronika ir Elektrotechnika (Electronics and Electrical Engineering),* no. 6, pp. 117–122, Jun. 2011.