A Comparison between Analytical Solutions for Lightning-Induced Voltage Calculation

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***Abstract*—An exact closed form solution for the calculation of lightning-induced voltages on overhead lines has been recently proposed by A. Andreotti *et al.* (2012). Predictions of this exact formulation are compared here to those based on approximate analytical solutions proposed in the literature.**

***Index Terms*—Lightning, single conductor line, induced voltage, analytical solution.**

I. INTRODUCTION

Distribution lines are very sensitive to nearby lightning strike events, and this can cause power quality problems. For this reason, a study on lightning-induced voltages shall be carried out.

In recent years, remarkable progress has been made in the studying lightning effects. Numerical approaches (e.g., [1]–[7]) allow very good modeling of the problem. Return-stroke current waveshape, influence of ground conductivity, surge arresters and other non-linearities can be taken into account. On the other hand, analytical solutions (e.g., [8]–[16]) are also very important in dealing with the design of the line. Furthermore, many computer codes use analytical solutions for the evaluation of lightning induced effects [17]. Finally, analytical solutions are not affected by instability problems [18].

A typical configuration that can be found in the calculation of lightning induced voltages is represented by a lossless, single conductor line, located over a perfectly-conducting ground and illuminated by a lightning field produced by a linearly-rising current which propagates according to the Transmission Line model [19], [20]. The configuration is shown in Fig. 1.

Approximate analytical solutions to this problem have been proposed by Chowdhuri and Gross [21], [22], Liew and Mar [23], Sekioka [24], and Hoidalen [10]. Andreotti *et al.* were able to find the exact solution [25]. In this paper, this exact solution will be compared to those proposed by other authors. The paper is organized as follows: in Section II, a survey of the solutions presented by other authors is carried out; in Section III, the Andreotti *et al.'s* solution is briefly reviewed; in Section IV, this solution is compared to the other solutions, and, finally, conclusions are drawn in Section V.

Manuscript received October 5, 2013; accepted December 14, 2013.

II. A SURVEY OF CLOSED-FORM SOLUTIONS

A. Chowdhuri-Gross Formula

Chowdhuri and Gross proposed a solution for a linearly-rising current of constant slope $\sim = I_0 / t_f$, where I_0 and t_f are, respectively, the peak value and the front time of the lightning current. This solution, obtained starting with a coupling model developed by the authors themselves [26], was first published in [21].

The solution was modified in [22] on the basis of the suggestions given by Cornfield [27]. We consider here the final expression specified for $x = 0$

$$v(0,t) = \frac{\text{«}_0 h}{4\text{®}} \cdot \sim \times$$
$$\times \left\{ b_0 \left[\ln\left(\frac{1}{b_0^2 y_0^2} \cdot \left[b_0 y_0^2 + \ {}^2 c^2 t^2 \left(1+ \ {}^2\right) - 2 \ {}^2 ct\text{¶} \right] \right) - \right.\right.$$
$$\left.\left. -2\ln\left(\frac{ct}{y_0}\right)\right] + \ln\left(\frac{f_1 \cdot f_3}{f_2 \cdot f_4}\right)\right\} \cdot u\left(t - t_0\right). \quad (1)$$

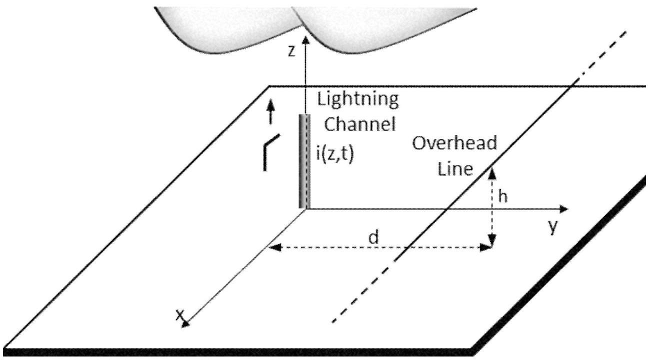

Fig. 1. A typical configuration for lightning-induced voltage calculations.

Symbols’ meaning can be found in [22]. Formula (1) refers to the rising part of the lightning current, it is therefore necessary to add an additional contribution to complete the overall current waveshape, which corresponds to its post-peak part (constant-level or drooping tail). As shown in Fig. 2, where t_t denotes the tail time of the lightning current, in the case of constant-level tail, this contribution consists of a time-delayed ramp of negative slope and the same magnitude as the positive ramp. For a drooping tail the second contribution has to be evaluated as explained in [25].

In summary, in the case of constant-level tail, the second contribution denoted by $v'(0,t)$ is given by

$$v'(0,t) = -v(0,t-t_f), \tag{2}$$

while for the drooping tail $v'(0,t)$ can be obtained as

$$v'(0,t) = -\frac{\sim^*}{\sim}\cdot v(0,t-t_f). \tag{3}$$

B. Liew-Mar Formula

The formula proposed by Liew and Mar was originally published in [23] and then revised in the discussion of [22]. In fact, the original formula had several typographical errors. The formula, specified for $x = 0$, reads

$$v(0,t) = \frac{\text{«}_0 h}{4\text{®}}\times\sim\times$$
$$\times\left\{\left[\ln\left(\frac{1}{b_0^2 y_0^2}\cdot\left[b_0 y_0^2 + \ ^2c^2t^2\left(1+\ ^2\right)-2\ ^2ct\text{¶}\right]\right)-\right.\right.$$
$$\left.-2\ln\left(\frac{ct}{y_0}\right)\right]-\left[\cosh^{-1}\left(\frac{u+p}{s}\right)-\cosh^{-1}\left(\frac{u_0+p}{s}\right)-\right.$$
$$\left.-\cosh^{-1}\left(\frac{z+p/q^2}{w}\right)+\cosh^{-1}\left(\frac{z_0+p/q^2}{w}\right)\right]+$$
$$\left.+2\ \left[\sinh^{-1}\left(\frac{ct}{y_0\sqrt{b_0}}\right)-\sinh^{-1}\left(\frac{ct_0}{y_0\sqrt{b_0}}\right)\right]\right\}\cdot u(t-t_0). \tag{4}$$

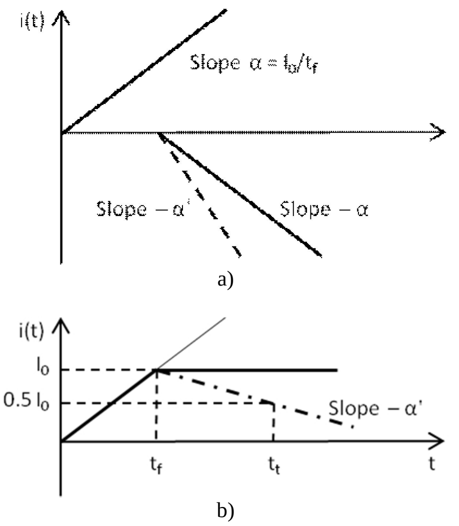

Fig. 2. Current waveshapes: ramp with constant or drooping tails.

Symbols’ meaning can be found in [22]. In this case too, (4) refers to the first part of the current. Another term must be considered, as done in the previous subsection.

C. Sekioka Formula

The formula proposed by Sekioka [24] was derived by using the so-called Rusck’s coupling model [9], [12]. The formula, for $x = 0$, reads

$$v(0,t) = \frac{\text{«}_0 h}{4\text{®}}\sim\left\{\ln\left[1+\left(\frac{\ }{d}\cdot\frac{c^2t^2-d^2}{ct+\text{¶}}\right)^2\right]+\right.$$
$$\left.+2\quad\ln\left(\frac{ct+\text{¶}}{d(1+\quad)}\right)\right\}u(t-t_0). \tag{5}$$

Symbols’ meaning can be found in [24]. As before, this formula gives only the contribution of the initial portion of the lightning current.

D. Hoidalen Formula

Hoidalen [10] proposed an approximate formula which allows one to evaluate the induced voltage along the line. This solution has been obtained by a numerical convolution of Rusck expression for the step current case.

As the solutions presented above, the Hoidalen solution is the sum of the two contributions, which account for both the rising part and the tail of the current. At $x = 0$, it reads

$$v(0,t) \approx A(0,t) - b\cdot A(0,t-t_f), \tag{6}$$

with

$$A(0,t) = \frac{I_m\cdot\Delta t}{I_0\cdot t_f}\cdot\left[\sum_{i=0}^{t/\Delta t-1} v_s(0,i\cdot\Delta t)+\frac{v_s(0,t)}{2}\right]\cdot u(t). \tag{7}$$

In this case, symbols’ meaning can be found in [10]. Note that the expression given in [10] refers to a finite length line. It can be easily extended to the case of an infinite length line.

III. ANDREOTTI ET AL. FORMULA

We will briefly review the formula proposed by Andreotti *et al.* [25] for the evaluation of the induced voltage at $x = 0$ in the case of a linearly-rising current (see Fig. 1).

This expression has been derived starting from the rigorous analytical solution presented by Andreotti at *et al.* for a step current [12] by using the Duhamel’s integral. For a linearly-rising current of constant slope $\sim = I_0/t_f$, the resulting expression is (see [25] for details)

$$v(0,t) = v_1(0,t) - v_2(0,t), \tag{8}$$

with

$$v_1(0,t) = -\frac{\text{«}_0}{4\text{®}}\sim\left\{\frac{d}{\ }\left[\operatorname{atan}\left(\frac{\text{a}}{d}\right)-\operatorname{atan}\left(\frac{\text{a}_0}{d}\right)-\right.\right.$$
$$\left.-\operatorname{atan}\left(\frac{d}{\sqrt{\text{a}^2+\text{¢}^2}}\right)+\operatorname{atan}\left(\frac{d}{\sqrt{\text{a}_0^2+\text{¢}^2}}\right)\right]-\sqrt{\text{a}^2+\text{¢}^2}+$$

$$+\sqrt{a_0^2+¢^2}+a\left\{\ln\left(a+\sqrt{a^2+¢^2}\right)+\right.$$

$$\left.+\frac{1}{\ }\left[\ln\left(-\ a+\sqrt{a^2+¢^2}\right)-1\right]\right\}-$$

$$-a_0\left\{\ln\left(a_0+\sqrt{a_0^2+¢^2}\right)+\right.$$

$$\left.+\frac{1}{\ }\left[\ln\left(-\ a_0+\sqrt{a_0^2+¢^2}\right)-1\right]\right\}u(t-t_0), \qquad (9)$$

where $«_0$ is the permeability of free space, is the ratio of the return stroke speed to c (the speed of light in free space), t is the time ($t=0$, return stroke inception), $a = \ ct-h$, $a_0 = \ ct_0-h$, $¢ = d/¥$, $t_0=r_0/c$, h is the line conductor height above ground, d is the horizontal distance between the lightning channel and the line conductor, $¥ = 1/\sqrt{1-\ ^2}$, $r_0=\sqrt{d^2+h^2}$, and $u(\cdot)$ is the Heaviside function. The expression of $v_2(0,t)$ can be derived from $v_1(0,t)$ by changing the sign of h.

Equations (8), (9) give the exact closed form expression for the linearly-rising portion of the lightning current. An additional contribution has to be added to obtain the voltage induced by the overall current waveshape, as done for previous models.

IV. COMPARISON OF THE EXACT CLOSED-FORM SOLUTION WITH OTHER SOLUTIONS PRESENTED IN LITERATURE

In this section, solution (8) will be compared to the other analytical solutions described in Section II.

For comparison purposes, we will assume a current at the base of the channel with a maximum value I_0 = 12 kA, a front time t_f = 0.5 μs and a drooping tail with t_t = 20 μs. This specific channel-base current was selected since it represents the best fit to a typical measured channel base current [10].

A. Chowdhuri-Gross Formula

In this paragraph, we will compare the induced voltage waveform obtained by using the Chowdhuri-Gross formula (1) with the one obtained by using our exact solution (8). In Fig. 3, the comparison is shown for a 10-meter height line placed at a distance d = 50 m away from the lightning channel; results have been obtained for the following values of the parameters, I_0 = 12 kA, = 0.4, t_f = 0.5 μs and t_t = 20 μs. The same comparison, but with d = 100 meters, is shown in Fig. 4. We point out that, since the expression proposed by Chowdhuri-Gross is given for a lightning channel of finite length (h_c), for comparison purposes we have assumed, for both graphs, $h_c \rightarrow \infty$ in (1). Furthermore, for the purpose of completeness, we have also considered a finite length channel having h_c = 3 km (see Fig. 5 and Fig. 6). The comparison clearly shows that the Chowdhuri-Gross's formula, similar to the case of step current analysed in [12], cannot be considered correct. We notice a polarity inversion, more pronounced when h_c is finite, which cannot be justified for the voltage induced by a linearly-rising current waveform with drooping tail in case of lossless ground.

B. Liew-Mar Formula

Now, we will compare results obtained by using our solution (8) and their counterparts obtained by using the Liew-Mar formula (4). In Fig. 7, the results obtained for h = 10 m, d = 50 m, I_0 = 12 kA, = 0.4, t_f = 0.5 μs and t_t = 20 μs are shown. In Fig. 8, the same comparison is shown, but for d = 100 m.

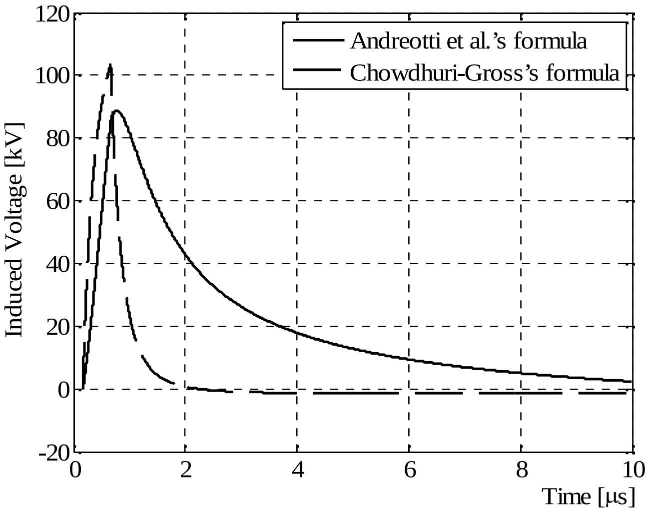

Fig. 3. Comparison between the voltage evaluated at x = 0 by means of Chowdhuri–Gross's and Andreotti *et al.*'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs, $h_c = \infty$).

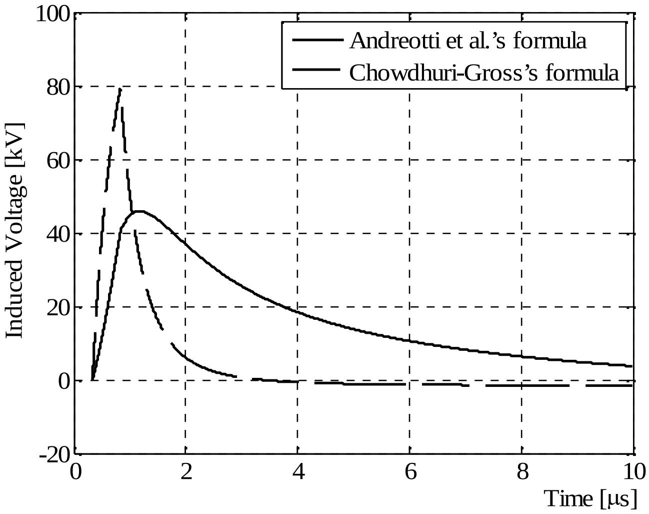

Fig. 4. Same as Fig. 3, but for d = 100 m.

As for the Chowdhuri-Gross's formula, the Liew-Mar solution also considers a finite channel length: for comparison we have assumed $h_c \rightarrow \infty$ in (4) for both graphs. To check the effects of a finite length channel, we have also considered the case h_c = 3 km (see Fig. 9 and

Fig. 10). As one can see from the comparison, the Liew-Mar formula predicts a lower peak value for d = 50 m, and a higher peak value for d = 100 m. In both cases, results obtained using (4) show a steeper front, a too rapid decay of the current tail, and a polarity inversion. The latter effect is seen both for finite and infinite lengths of the lightning channel.

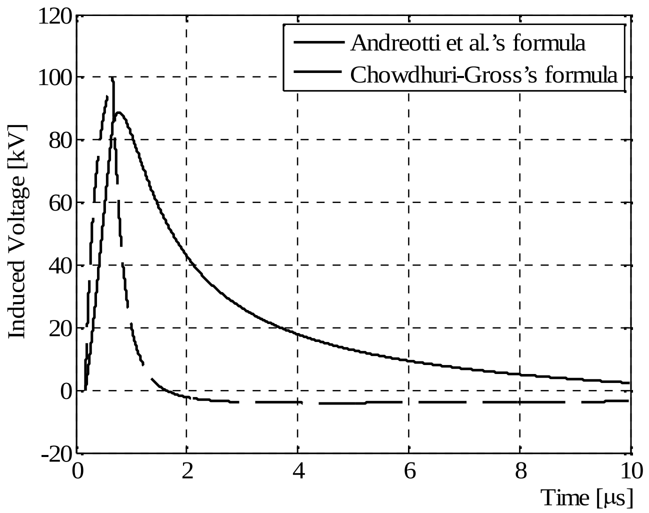

Fig. 5. Comparison between the voltage evaluated at x = 0 by means of Chowdhuri–Gross's and Andreotti *et al*.'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs, h_c = 3 km).

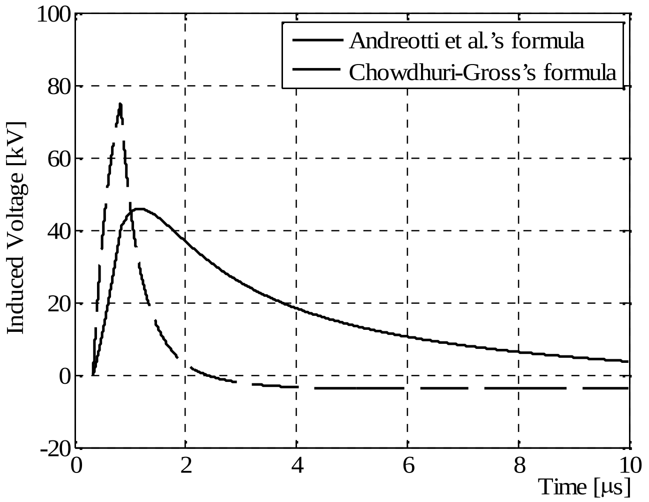

Fig. 6. Same as Fig. 5, but for d = 100 m.

We conclude that the Liew-Mar solution cannot be considered correct.

C. Sekioka Formula

By comparing the results obtained using solution (8) and their counterparts obtained by means of the Sekioka formula (5), calculated for the same values of parameters proposed in the previous paragraphs (h = 10 m, d = 50, 100 m, I_0 = 12 kA, = 0.4, t_f = 0.5 μs and t_t = 20 μs), one can observe that the results are practically the same (overlapped). Some minor differences can be spotted by zooming in the graphs, as shown in Fig. 11 for the case of d = 50 m.

We can therefore conclude that the Sekioka's formula is consistent with the exact solution and can be considered a useful tool for the analysis of power distribution lines, for which the height is relatively small. For higher lines, such as transmission lines, small differences between predictions of (8) and (5) can be detected. However, the Sekioka formula can still be considered a suitable tool for this kind of lines. In Fig. 12, a 30-meter height line is considered.

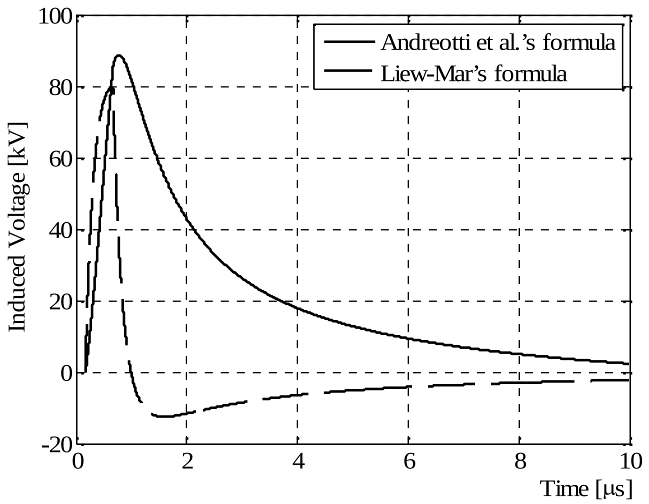

Fig. 7. Comparison between the induced voltage evaluated at x = 0 by means of Liew-Mar's and Andreotti *et al*.'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs, h_c = ∞).

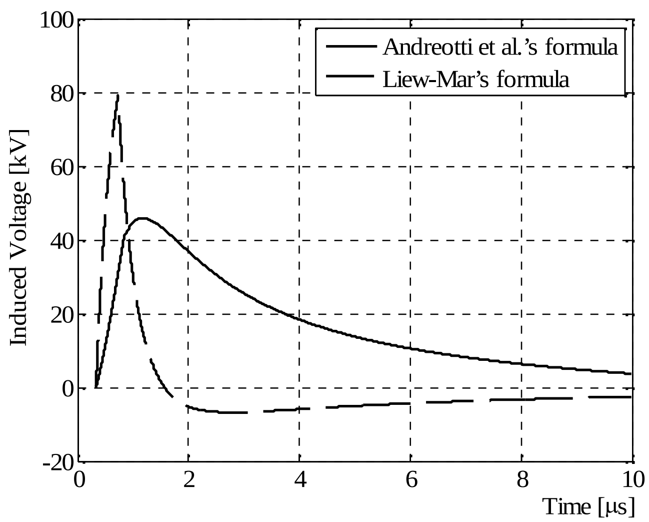

Fig. 8. Same as Fig. 7, but for d = 100 m.

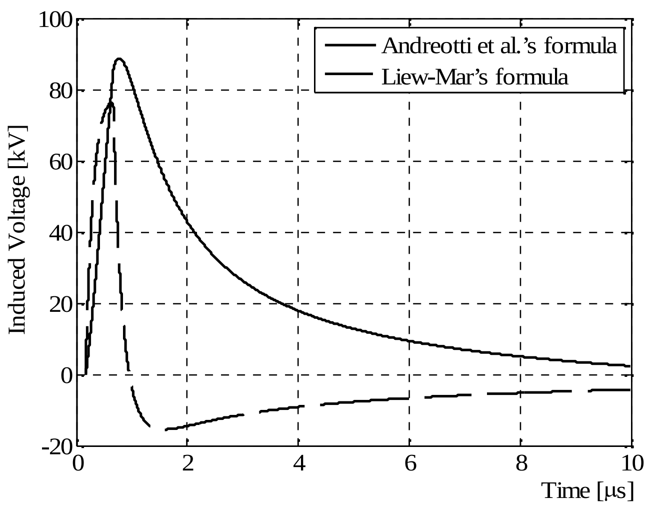

Fig. 9. Comparison between the induced voltage evaluated at x = 0 by means of Liew-Mar's and Andreotti *et al*.'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs, h_c = 3 km).

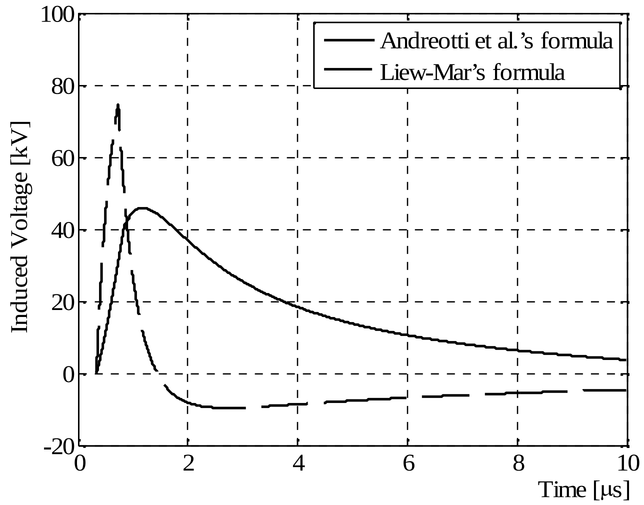

Fig. 10. Same as Fig. 9, but for d = 100 m.

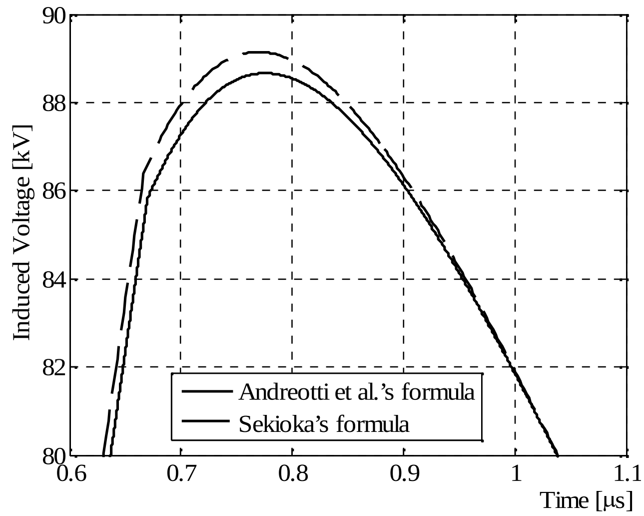

Fig. 11. Comparison between the voltage evaluated at x = 0 by means of Sekioka's and Andreotti *et al*.'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs): magnification of the 0.6÷1.1 μs time interval.

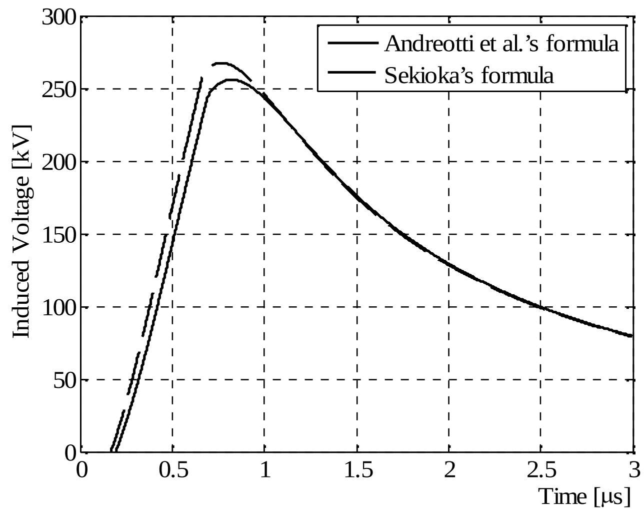

Fig. 12. Comparison between the voltage evaluated at x = 0 by means of Sekioka's and Andreotti *et al*.'s formulas (h = 30 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs).

D. Hoidalen Formula

A comparison between (8) and Hoidalen formula (6) is shown in Fig. 13 and Fig. 14, where a 10 m and a 30 m lines have been considered, respectively. As in the case of Sekioka's expression, practically no differences are observed for a 10 m line (the graph was zoomed in to show some minor differences), whereas for the 30-m line, even if differences can be seen, Hoidalen's formula can still be considered a suitable approximation of the exact solution.

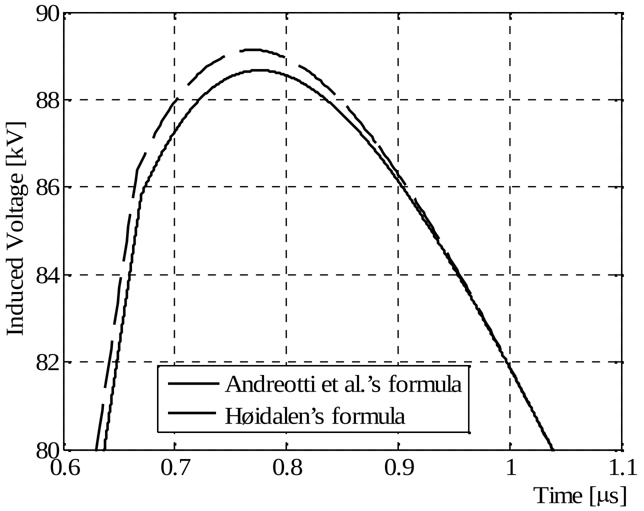

Fig. 13. Comparison between the voltage evaluated at x = 0 by means of Hoidalen's and Andreotti *et al*.'s formulas (h = 10 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs): magnification of the 0.6÷1.1 μs time interval.

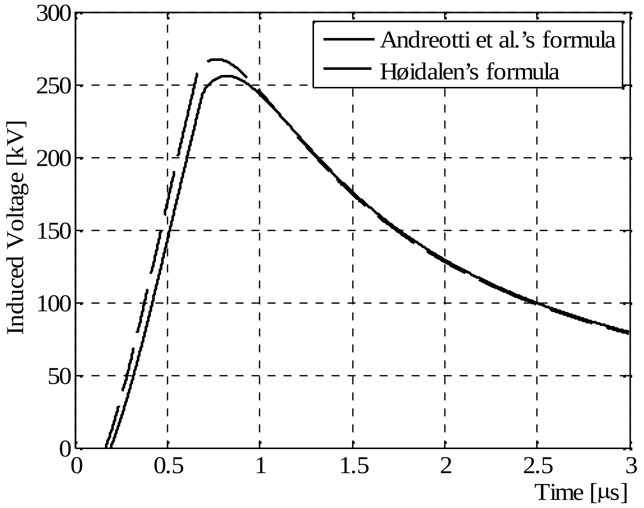

Fig. 14. Comparison between the voltage evaluated at x = 0 by means of Hoidalen's and Andreotti *et al*.'s formulas (h = 30 m, d = 50 m, I_0 = 12 kA, β = 0.4, t_f = 0.5 μs, t_t = 20 μs).

V. CONCLUSIONS

In this work, the rigorous analytical solution presented by Andreotti *et al.* [25] for the calculation of voltages induced on an overhead conductor located above a perfectly-conducting ground by a linearly-rising lightning current waveform has been reviewed and predictions of this formulation have been compared to those given by other analytical (approximate) solutions proposed in the literature.

The comparison shows that both the Chowdhuri-Gross and the Liew-Mar formulas predict results which are in disagreement with the Andreotti *et al.* solution. In particular, they predict incorrect peak values, steeper fronts, too rapid decays of the current tail, and physically unexplainable polarity inversions. Hence, they cannot be considered correct. Conversely, both the Sekioka solution and the Hoidalen's formula are consistent with the exact analytical solution, and can be considered suitable for lightning induced overvoltage analysis of distribution lines, i.e., lines with relatively small height. For such lines, differences between Andreotti *et al.'s* formula and both the Sekioka and the Hoidalen's expressions are within the 0.6 %.

REFERENCES

[1] F. Rachidi, C. A. Nucci, M. Ianoz, "Transient analysis of multiconductor lines above a lossy ground", *IEEE Trans. Power Del.*, vol. 14, no. 1, pp. 294–302, 1999. [Online]. Available: http://dx.doi.org/10.1109/61.736741

[2] A. Andreotti, A. Del Pizzo, R. Rizzo, L. Verolino, "Lightning induced effects on lossy multiconductor power lines with ground wires and non-linear loads - Part I: model", *Przeglad Elektrotechniczny (Electrical Review)*, vol. 88, no. 9b , pp. 301–304, 2012.

[3] R. Rizzo, A. Andreotti, A. Del Pizzo, L. Verolino, "Lightning induced effects on lossy multiconductor power lines with ground wires and non-linear loads - Part II: simulation results and experimental validation", *Przeglad Elektrotechniczny (Electrical Review)*, vol. 88, no. 9b, pp. 305–308, 2012.

[4] G. Diendorfer, "Induced voltage on an overhead line due to nearby lightning", *IEEE Trans. Electromagn. Compat*., vol. 32, no. 4, pp. 292–299, 1990. [Online]. Available: http://dx.doi.org/10.1109/15.59889

[5] H. K. Hoidalen, J. Slebtak, T. Henriksen, "Ground effects on induced voltages from nearby lightning", *IEEE Trans. Electromagn. Compat.*, vol. 32, no. 4, pp. 292–299, 1990.

[6] A. Andreotti, C. Petrarca, V. A. Rakov, L. Verolino, "Calculation of voltages induced on overhead conductors by nonvertical lightning channels", *IEEE Trans. Electromagn. Compat.*, vol. 54, no. 4, pp. 860–870, 2012. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2011.2174995

[7] A. Andreotti, U. De Martinis, C. Petrarca, V. A. Rakov, L. Verolino, "Lightning electromagnetic fields and induced voltages: Influence of channel tortuosity", in *Proc. XXXth URSI General Assembly and Scientific Symposium*, Istanbul, Turkey, 2011, pp. 1–4. [Online]. Available: http://dx.doi.org/10.1109/URSIGASS.2011.6050702

[8] J. O. S. Paulino, C. F. Barbosa, W. C. Boaventura, "Effect of the surface impedance on the induced voltages in overhead lines from nearby lightning", *IEEE Trans. Electromagn. Compat.*, vol. 53, no. 3, pp. 749–754, 2011. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2011.2155661

[9] S. Rusck, "Induced lightning overvoltages on power transmission lines with special reference to the overvoltage protection of low voltage networks", *Trans. Royal Inst. Technol.*, no. 120, pp. 1–118, 1958.

[10] H. K. Hoidalen, "Analytical formulation of lightning-induced voltages on multiconductor overhead lines above lossy ground", *IEEE Trans. Electromagn. Compat.*, vol. 45, no. 1, pp. 92–100, 2003. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2002.804772

[11] F. Napolitano, "An analytical formulation of the electromagnetic field generated by lightning return strokes", *IEEE Trans. Electromagn. Compat*., vol. 53, no. 1, pp. 108–113, 2011. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2010.2065810

[12] A. Andreotti, D. Assante, F. Mottola, L. Verolino, "An exact closed-form solution for lightning-induced overvoltages calculations", *IEEE Trans. Power Del.*, vol. 24, no. 3, pp. 1328–1343, 2009. [Online]. Available: http://dx.doi.org/10.1109/TPWRD.2008.2005395

[13] D. Assante, A. Andreotti, L. Verolino, "Considerations on the characteristic impedance of periodically grounded multiconductor transmission lines", *Int. Symposium on Electromagnetic Compatibility (EMC Europe 2012)*, Rome, Italy, 2012.

[14] C. F. Barbosa, J. O. S. Paulino, "A time-domain formula for the horizontal electric field at the earth surface in the vicinity of lightning", *IEEE Trans. Electromagn. Compat*., vol. 52, no. 3, pp. 640–645, 2010. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2010.2047647

[15] *IEEE Guide for Improving the Lightning Performance of Electric Power Overhead Distribution Lines*, IEEE Standard 1410, 2010.

[16] P. Chowdhuri, "Parametric effects on the induced voltages on overhead lines by lightning strokes to nearby ground", *IEEE Trans. Power Del.*, vol. 4, no. 2, pp. 1185–1194, 1989. [Online]. Available: http://dx.doi.org/10.1109/61.25602

[17] H. K. Hoidalen, "Calculation of lightning-induced overvoltages using models", in *Proc. Int. Conf. Power Syst. Trans.*, Budapest, Hungary, 2003, pp. 7–12.

[18] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*. Philadelphia, PA: SIAM, 2002. [Online]. Available: http://dx.doi.org/10.1137/1.9780898718027

[19] M. A. Uman, D. K. McLain, "Magnetic field of the lightning return stroke", *J. Geophys. Res*., vol. 74, no. 28, pp. 6899–6910, 1969. [Online]. Available: http://dx.doi.org/10.1029/JC074i028p06899

[20] A. Andreotti, D. Assante, V. A. Rakov, L. Verolino, "Electromagnetic coupling of lightning to power lines: Transmission-line approximation versus full-wave solution", *IEEE Trans. Electromagn. Compat*., vol. 53, no. 2, pp. 421–428, 2011. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2010.2091682

[21] P. Chowdhuri, E. T. B. Gross, "Voltage surges induced on overhead lines by lightning strokes", in *Proc. Inst. Electr. Eng.*, 1967, vol. 114, no. 12, pp. 1899–1907. [Online]. Available: http://dx.doi.org/10.1049/piee.1967.0363

[22] P. Chowdhuri, "Analysis of lightning-induced voltages on overhead lines", *IEEE Trans. Power Del.*, vol. 4, no. 1, pp. 479–492, 1989. [Online]. Available: http://dx.doi.org/10.1109/61.19238

[23] A. C. Liew, S. C. Mar, "Extension of the Chowdhuri-Gross model for lightning induced voltage on overhead lines", *IEEE Trans. Power Syst.*, vol. 1, no. 2, pp. 240–247, 1986. [Online]. Available: http://dx.doi.org/10.1109/TPWRD.1986.4307956

[24] S. Sekioka, "An equivalent circuit for analysis of lightning-induced voltages on multiconductor system using an analytical expression", in *Proc. Int. Conf. Power Syst. Trans.*, Montreal, Canada, 2005.

[25] A. Andreotti, A. Pierno, V. A. Rakov, L. Verolino, "Analytical formulations for lightning-induced voltage calculations", *IEEE Trans. Electromagn. Compat.*, vol. 55, no. 1, pp. 109–123, 2013. [Online]. Available: http://dx.doi.org/10.1109/TEMC.2012.2205001

[26] V. Cooray, "Calculating lightning-induced overvoltages in power lines. A comparison of two coupling models", *IEEE Trans. Electromag. Compat.*, vol. 36, no. 3, pp. 179–182, 1994. [Online]. Available: http://dx.doi.org/10.1109/15.305462

[27] G. Cornfield, "Voltage surges induced on overhead lines by lightning strokes", in *Proc. IEE*, vol. 119, no. 7, 1972, pp. 893–894.