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Linear Phase Two-Dimensional FIR Digital Filter Functions Generated by applying Christoffel-Darboux Formula for Orthonormal Polynomials

D. G. Ciric,V. D. Pavlovic
Faculty of Electronic Engineering, University of Niš,
Aleksandra Medvedeva 14, P.O Box 73, 18000 Niš, Republic of Serbia, phones: +381 18 529 301, +381 18 529 206,
e-mails: dejan.ciric@elfak.ni.ac.rs, vlastimir.pavlovic@elfak.ni.ac.rs

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Introduction

Filter theory represents one of the strictest disciplines with the possibilities of applications in various frequency ranges and technologies [1–4]. In this theory, successful applications of powerful orthogonal polynomials are well-known [4–7]. A number of problems in various scientific and technical areas have been solved by applying the classical Christoffel-Darboux formula for all classic orthogonal polynomials [8, 9]. New class explicit filter functions for continuous signals generated by the classical Christoffel-Darboux formula for classical Jacobi and Gegenbauer orthonormal polynomials are described in detail [10, 11]. On the other hand, there have been a number of attempts to solve the complex problem of generating linear phase two-dimensional finite impulse response (FIR) digital filters of lower order, e.g. [12]. They are based on either a transformation of one-dimensional FIR filters or direct application of the approximation techniques in two dimensions.

Further generalization of the previous research [10, 11] in two dimensions is presented in this paper. The global Christoffel-Darboux formula for four orthonormal polynomials on two equal finite segments for generating filter functions is proposed here in a compact explicit form. A new class of the linear phase two-dimensional FIR digital filters generated by the proposed formula is given.

Mathematical background

Let $P_r(x)$ and $Q_m(x)$ be two sets of orthogonal polynomials, where x is a real variable, r and m are the orders of continuous non-periodical polynomials on a finite interval $-a \le x \le b$ with respect to the non-negative continuous weight functions, $w_1(x)$ and $w_2(x)$, respectively, and orthogonality defined by:

$$\int_{-a}^{b} w_1(x) P_r(x) P_k(x) dx = 0 \quad r \ne k;\ r, k = 0,1,2,3,\dots \quad (1)$$

$$\int_{-a}^{b} w_2(x) Q_m(x) Q_k(x) dx = 0 \quad m \ne k;\ m, k = 0,1,2,3,\dots \quad (2)$$

For the polynomial $P_r(x)$, r-th order norm, $h_1(r)$, is given by

$$h_1(r) = \int_{-a}^{b} w_1(x) \left(P_r(x)\right)^2 dx \quad r = 1,2,3,\dots, \quad (3)$$

while m-th order norm, $h_2(m)$, for the polynomial $Q_m(x)$, is

$$h_2(m) = \int_{-a}^{b} w_2(x) \left(Q_m(x)\right)^2 dx \quad m = 1,2,3,\dots \quad (4)$$

Besides, let $R_r(y)$ and $S_m(y)$ be other two sets of polynomials, where y is a real variable, r and m are the orders of continuous non-periodical polynomials on a finite interval $-c \le y \le d$ with respect to the non-negative continuous functions, $w_3(y)$ and $w_4(y)$, respectively, and orthogonality defined by:

$$\int_{-c}^{d} w_3(y) R_r(y) R_k(y) dy = 0 \quad r \ne k;\ r, k = 0,1,2,3,\dots \quad (5)$$

$$\int_{-c}^{d} w_4(y) S_m(y) S_k(y) dy = 0 \quad m \ne k;\ m, k = 0,1,2,3,\dots \quad (6)$$

For the mentioned polynomials, $R_r(y)$ and $S_m(y)$, r-th order and m-th order norms, $h_3(r)$ and

$h_4(m)$, respectively, are:

$$h_3(r)=\int_{-c}^{d} w_3(y)\left(R_r(y)\right)^2 dy \quad r=1,2,3,\ldots \tag{7}$$

$$h_4(m)=\int_{-c}^{d} w_4(y)\left(S_m(y)\right)^2 dy \quad m=1,2,3,\ldots \tag{8}$$

The finite (summed from zero to n-th component) global Christoffel-Darboux formula for two same order orthogonal polynomials with x as a variable, $P_r(x)$ and $Q_m(x)$, $r, m=1,2,\ldots,n$ (n is the order of continual orthogonal polynomials), on the equal finite segment $[a,b]$, and for two same order orthogonal polynomials with y as a variable, $R_r(y)$ and $S_m(y)$, on the equal finite segment $[c,d]$, is proposed here in the following explicit compact representative form of orthonormal components:

$$\begin{gathered}\frac{P_0(x)Q_0(x)R_0(y)S_0(y)}{\sqrt{h_1(0)h_2(0)h_3(0)h_4(0)}}+ \\ +\frac{P_1(x)Q_1(x)R_1(y)S_1(y)}{\sqrt{h_1(1)h_2(1)h_3(1)h_4(1)}}+\ldots \\ +\frac{P_k(x)Q_k(x)R_k(y)S_k(y)}{\sqrt{h_1(k)h_2(k)h_3(k)h_4(k)}}+\ldots \\ +\frac{P_n(x)Q_n(x)R_n(y)S_n(y)}{\sqrt{h_1(n)h_2(n)h_3(n)h_4(n)}}\end{gathered}, \tag{9}$$

or

$$\sum_{r=0}^{n}\frac{P_r(x)Q_r(x)R_r(y)S_r(y)}{\sqrt{h_1(r)h_2(r)h_3(r)h_4(r)}}. \tag{10}$$

By standard technique, the previous formula can be mapped into the new domains, analogue, s, and digital, z, [13–15]. For example, in the z_1 (or z_2) domain, the following relations are always valid:

$$T_k \to \cos\left(k\omega_1\right) \to \left(z_1^{k}+z_1^{-k}\right)/2, \tag{11}$$

$$U_k \to \sin\left(k\omega_1\right) \to \left(z_1^{k}-z_1^{-k}\right)/(2j), \tag{12}$$

where T_k and U_k are the orthogonal Chebyshev polynomials of the first kind and second kind, respectively. Alternatively, the mapping into the z_1 (or z_2) domain can be represented by

$$(x)^k=\left(T_1(x)\right)^k \to \left(\cos(\omega_1)\right)^k \to \left[\left(z_1^{+1}+z_1^{-1}\right)/2\right]^k. \tag{13}$$

The third way of mapping is given by the following example:

$$\begin{gathered}x^{10}=\frac{1}{512}\left[+126T_0(x)+210T_2(x)+120T_4(x)+\right. \\ \left.+45T_6(x)+10T_8(x)+T_{10}(x)\right]\to \\ \to\frac{1}{512}\left[+126+210\cos(2\omega_1)+120\cos(4\omega_1)+\right. \\ \left.+45\cos(6\omega_1)+10\cos(8\omega_1)+\cos(10\omega_1)\right]\to \\ \to\frac{1}{1024}\left[+126+210(z_1^{+2}+z_1^{-2})+120(z_1^{+4}+z_1^{-4})+\right. \\ \left.+45(z_1^{+6}+z_1^{-6})+10(z_1^{+8}+z_1^{-8})+(z_1^{+10}+z_1^{-10})\right],\end{gathered} \tag{14}$$

$$\begin{gathered}x^{9}=\frac{1}{256}\left[+126T_1(x)+84T_3(x)+\right. \\ \left.36T_5(x)+9T_7(x)+T_9(x)\right]\to \\ \to\frac{1}{256}\left[+126\cos(\omega_1)+84\cos(3\omega_1)+\right. \\ \left.+36\cos(5\omega_1)+9\cos(7\omega_1)+\cos(9\omega_1)\right]\to \\ \to\frac{1}{512}\left[+126(z_1^{+1}+z_1^{-1})+84(z_1^{+3}+z_1^{-3})+\right. \\ \left.+36(z_1^{+5}+z_1^{-5})+9(z_1^{+7}+z_1^{-7})+(z_1^{+9}+z_1^{-9})\right].\end{gathered} \tag{15}$$

Filter function

A linear phase two-dimensional FIR filter of $N\times N$-order is defined by

$$H\left(z_1, z_2\right)=K\sum_{r=0}^{N}\sum_{k=0}^{N} b(r,k)\, z_1^{-r} z_2^{-k}, \tag{16}$$

where K is the gain constant and $b(r,k)$ are the filter coefficients that are real numbers. Square of the filter frequency response can be represented by

$$H\left(z_1, z_2\right)H\left(\overline{z_1}, \overline{z_2}\right), \text{ for } z_1\to e^{j\omega_1}, z_2\to e^{j\omega_2} \tag{17}$$

or alternatively in absolute units and dBs, respectively:

$$\left|H\left(e^{j\omega_1}, e^{j\omega_2}\right)\right|^2= =\left|H\left(e^{j\omega_1}, e^{j\omega_2}\right)H\left(e^{-j\omega_1}, e^{-j\omega_2}\right)\right|, \tag{18}$$

$$20\log\left|H\left(e^{j\omega_1}, e^{j\omega_2}\right)\right|. \tag{19}$$

New class of two-dimensional FIR filter functions

Applying the proposed formula, Eq. (9), a new class of two-dimensional FIR filter functions is obtained as

$$H\left(z_1, z_2\right)=K\sum_{r=0}^{N}\left(\frac{P_r\left(\frac{z_1^{+1}+z_1^{-1}}{2}\right)Q_r\left(\frac{z_1^{+1}+z_1^{-1}}{2}\right)}{1}\times\frac{R_r\left(\frac{z_2^{+1}+z_2^{-1}}{2}\right)S_r\left(\frac{z_2^{+1}+z_2^{-1}}{2}\right)}{\sqrt{h_1(r)h_2(r)h_3(r)h_4(r)}}\right). \tag{20}$$

For the linear phase two-dimensional symmetric FIR digital filters generated by the proposed approximation technique, the following simetries are valid:

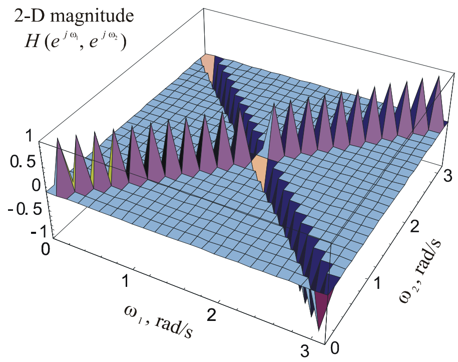
$$H(z_1, z_2) = H(z_2, z_1), \tag{21}$$

$$H(z_1, z_2) = H(-z_1, -z_2). \tag{22}$$

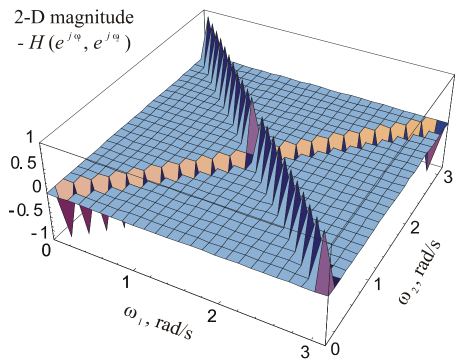
The linear phase function of the two-dimensional symmetric FIR digital filter defined by Eq. (20) has the following form for $z_1 \to e^{j\omega_1}$ and $z_2 \to e^{j\omega_2}$

$$e^{-j2N(\omega_1+\omega_2)}. \tag{23}$$

The two-dimensional frequency response of this filter for the parameters $a = c = -\pi$ and $b = d = +\pi$ in absolute units and dBs as well as the contour plot is given in Fig. 1 –Fig. 3. The view from above of the frequency response is presented in Fig. 1(a), 2(a) and 3(a), while the view from below (the corresponding response multiplied by -1) is presented in Fig. 1(b), 2(b) and 3(b).

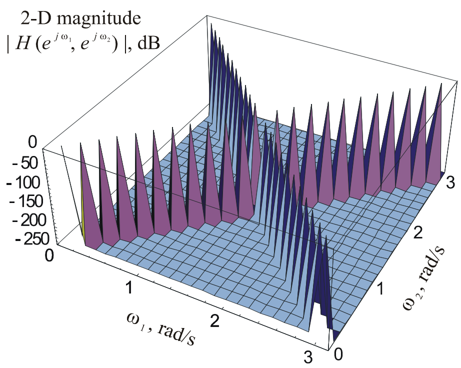


1(a)

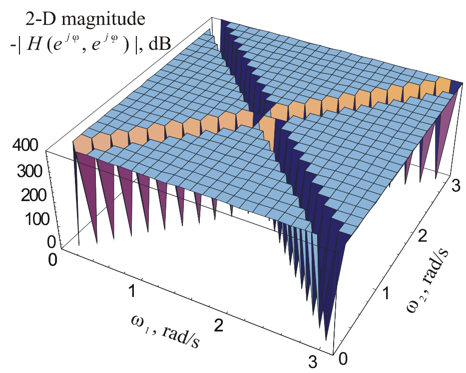


1(b)

Fig. 1. A three-dimensional (3-D) plot of two-dimensional frequency response of the linear phase two-dimensional FIR digital filter designed by the proposed formula: (a) view from above, (b) view from below

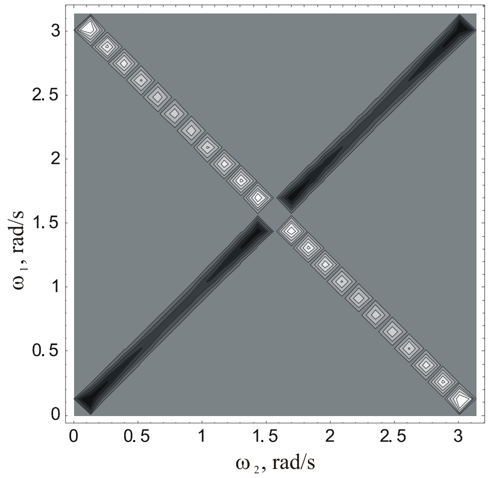


2(a)

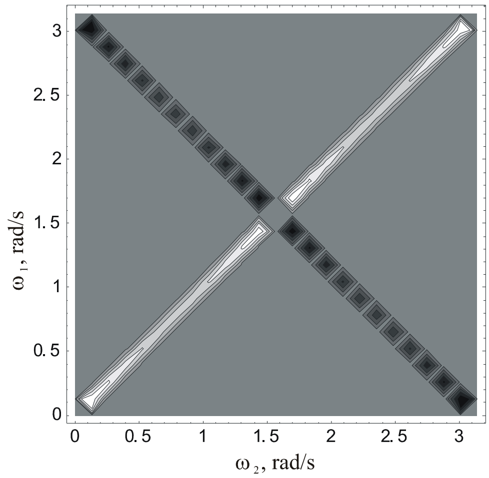


2(b)

Fig. 2. 3-D plot of two-dimensional frequency response (in dBs) of the linear phase two-dimensional FIR digital filter designed by the proposed formula: (a) view from above, (b) view from below



(a)



(b)

Fig. 3. Two-dimensional frequency response (contour plot) of the linear phase two-dimensional FIR digital filter generated by the proposed formula: (a) view from above, (b) view from below

Conclusions

This paper presents an original approach to linear phase two-dimensional FIR digital filter design yielding significant improvements. The global Christoffel-Darboux formula for four orthonormal polynomials on two equal finite segments is proposed in a compact explicit form. The proposed formula represents a superior identity for solving extremely complex and always actual problem of linear phase two-dimensional filter design. The formula can be most directly applied in generating two-dimensional filter functions. It enables efficient design of high order filters. The filters designed in this way are highly selective, and all parasitic effects are suppressed. These filters can be applied in various areas including telecommunications where they can be of special interest. Three-dimensional frequency response (and corresponding contour plot) of a new class linear phase two-dimensional FIR digital filter is presented illustrating the advantages of the proposed approach.

Acknowledgements

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Global Christoffel-Darboux formula for four orthonormal polynomials on two equal finite segments for generating linear phase two-dimensional finite impulse response (FIR) digital filter functions in a compact explicit representative form is proposed in this paper. The formula can be most directly applied for solving mathematically the approximation problem of a filter function of even and odd order. An example of a new class extremely economic linear phase two-dimensional FIR digital filter without multipliers obtained by the proposed approximation technique is presented. The generated linear phase two-dimensional FIR filter functions have two symmetries, that is, the following relations are valid: $H(z_1, z_2) = H(z_2, z_1)$ and $H(z_1, z_2) = H(-z_1, -z_2)$. Ill. 3, bibl. 15 (in English; abstracts in English and Lithuanian).

D. G. Ciric,V. D. Pavlovic. Tiesinės fazės dvimačio baigtinio impulso atsako skaitmeninio filtro funkcijos, gautos taikant Christofelio ir Darboux‘o formulę ortonormuotiesiems polinomams // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2012. – Nr. 4(120). – P. 39–42.

Pasiūlytos bendros Christofelio ir Darboux‘o formulės keturiems ortonormuotiesiems polinomams dviejų lygių baigtiniuose segmentuose generuojant tiesinės fazės dvimačio baigtinio impulso atsako (BIA) skaitmeninio filtro funkcijas. Formulė gali būti tiesiogiai taikoma sprendžiant filtro funkcijos matematinio aproksimavimo uždavinį. Taikant siūlomą aproksimavimo metodą, gautas labai ekonomiškas naujos klasės tiesinės fazės dvimatis BIA skaitmeninis filtras. Sugeneruotos funkcijos turi dvi simetrijas, todėl galioja tokios lygybės: $H(z_1, z_2) = H(z_2, z_1)$ ir $H(z_1, z_2) = H(-z_1, -z_2)$. Il. 3, bibl. 15 (anglų kalba; santraukos anglų ir lietuvių k.).