http://dx.doi.org/10.5755/j01.eee.20.10.4454 ELEKTRONIKA IR ELEKTROTECHNIKA, ISSN 1392-1215, VOL. 20, NO. 10, 2014

Complexity Analysis of the Piece-Wise Affine Approximation for the Car on the Nonlinear Hill Model Related to Discrete–Time, Minimum Time Control Problem

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***Abstract*—The main aim of the paper is to analyse effectiveness of the approximation method from the nonlinear discretised into discrete piece-wise affine model. Accuracy and numerical complexity of the piece-wise affine control system grow with the number of polyhedral partitions, that describe the system. The model with state dependent nonlinearity can be effectively approximated by proposed secant piece-wise linear approximation. The effectiveness of the proposed method is evaluated on the car on the nonlinear hill model.**

***Index Terms*—Predictive control; optimisation, piecewise linear approximation, discrete-time systems.**

I. INTRODUCTION

In the last decade growing attention was paid to modelling and methods for piece-wise affine (PWA) systems [1]–[5]. Powerful tools for control of PWA systems are implemented, i.e. MPT toolbox for Matlab [6]. From the practical point of view most of the PWA models are only approximation of the real nonlinear systems [5], [7]–[9]. Complexity of the PWA model is directly connected with the number of polyhedral partitions, describing the system. In general the more polyhedral partitions have the model; the more accurate is PWA approximation of the nonlinear system. On the other hand, computation time and numerical complexity of the control problem rapidly grows with the number of partitions [1].

Usually controller computed for the approximated model (e.g. PWA) operates on the nonlinear plant. It may be expected that the more accurate is the PWA approximation; the better is the closed loop performance of the system with nonlinear plant and controller tuned for PWA model. Moreover it may be supposed that there exists some minimal PWA approximation error that guarantees some specific performance of the control system with nonlinear plant.

The main aim of the paper is to propose and evaluate method for piecewise affine approximation of nonlinear system. Among many existing papers regarding control of PWA system and nonlinear system, there is a gap involving accuracy and complexity analysis of PWA approximated models in control of nonlinear system. Significant impact of the paper is verification how the number of partitions in the approximated PWA model results in control performance for the nonlinear system. The investigation is made on the basis of the assumed model – car on the nonlinear hill, where the control objective is to drive the car from initial conditions to the origin, in minimum time, subject to limited force.

Manuscript received May 24, 2014; accepted November 11, 2014.

II. SYSTEM DESCRIPTION AND CONTROL PROBLEM FORMULATION

The system under consideration is second order, discrete-time, nonlinear system, which can be described by the following state space model:

$$x_1(k+1) = x_1(k) + x_2(k)\Delta_T, \tag{1}$$

$$x_2(k+1) = -h(x_1)\Delta_T + x_2(k)(1-b\Delta_T) + a\Delta_T u(k), \tag{2}$$

where $h(x)$ is nonlinear continuous, bounded and real valued function, Δ_T is sampling period, $a,b \in \mathbb{R}$ are scalar parameters, $k \in \mathbb{Z}$ is discrete time, $\mathbf{x}(k) = \left[x_1(k)\ x_2(k)\right]^T$, $\mathbf{x}(k) \in \mathcal{X} \subset \mathbb{R}^2$, $x_1(k) \in \mathcal{X}_1$, $x_2(k) \in \mathcal{X}_2$, $u(k) \in \mathcal{U} \subset \mathbb{R}$. $\mathcal{X},\mathcal{U}$ are convex, compact (i.e. bounded and closed) sets containing the origin in their interior.

III. NONLINEAR FUNCTION PWA APPROXIMATION

In order to compute off-line controller the nonlinear function $h(x_1)$ must be substituted by their piece-wise affine (PWA) approximation $h_a^n(x_1)$, where indexes *a*, *n* denotes PWA approximation with n polyhedral partitions in the PWA model

$$h_a^n(x_1) = \begin{cases} c_1 x_1 + d_1, & l_0^x \le x_1 \le l_1^x, \\ c_2 x_1 + d_2, & l_1^x \le x_1 \le l_2^x, \\ \vdots & \vdots \\ c_n x_1 + d_n, & l_{n-1}^x \le x_1 \le l_n^x, \end{cases} \tag{3}$$

where $l_0^x = \inf(\mathcal{X}_1)$, $l_n^x = \sup(\mathcal{X}_1)$ are respectively lower and upper bound of x_1 over set $\mathcal{X}$, $\Delta_{x1} = l_n^x - l_0^x$ is the interval of variable x_1 over set $\mathcal{X}$.

Variables l_i define relative intersections of polyhedral partitions on the x_1 axis such that $0 = l_0 \le l_1 \le ... \le l_{n-1} \le l_n = 1$.

Absolute locations of intersections on the x_1 axis can be computed from: $l_i^x = l_i \Delta_{x1} + l_0^x$ for all $i = 0,1,...,n$.

The simplest PWA approximation is fixed length partitions approximation. In that case locations of intersections on the x_1 axis can be computed from: $l_i = i/n$ for all $i = 0,1,...,n$. Accuracy of the approximation is strongly connected with the number of polyhedral partitions. The fixed length partitions approximation provide acceptable accuracy only for large number of sectors. Although a large class of nonlinear functions can be effectively approximated by PWA models with reduced partitions number by proposed below secant approximation.

Secant Approximation of the Nonlinear Function

For secant approximation of the nonlinear function $h(x_1)$ absolute location of intersections on the *h* axis are equal to $p_i = h\left(l_i^x\right)$ for all $i = 0,1,...,n$.

Coefficients c_i, d_i for each partition in (3) can be computed in the following way

$$c_i = \frac{p_i - p_{i-1}}{l_i^x - l_{i-1}^x}, \tag{4}$$

and

$$d_i = \frac{p_{i-1} l_i^x - p_i l_{i-1}^x}{l_i^x - l_{i-1}^x}. \tag{5}$$

In order to ensure closed loop stability linear quadratic regulator (LQR) will be applied around the origin. The linear regulator can be chosen and applied only for linear systems, so the model must be linear in the origin, i.e. $d_i = 0$ for partition containing the origin in their interior $l_{i-1}^x \le 0 \le l_i^x$ and the coefficient d_i finally takes following form:

$$\begin{cases} d_i = \dfrac{p_{i-1} l_i^x - p_i l_{i-1}^x}{l_i^x - l_{i-1}^x}, & l_{i-1}^x > 0 \vee l_i^x < 0, \\ d_i = 0, & l_{i-1}^x \le 0 \le l_i^x. \end{cases} \tag{6}$$

Approximation error can be computed from the state difference between PWA and nonlinear system

$$\mathbf{x}(k+1) - \mathbf{x}^a(k+1) = \begin{bmatrix} 0 \\ \left(h_a^n(x_1) - h(x_1)\right)\Delta_T \end{bmatrix}. \tag{7}$$

The error function can be written in the following form

$$e^n(x_1) = \left(h_a^n(x_1) - h(x_1)\right)\Delta_T. \tag{8}$$

The secant PWA approximation optimization problem can be formulated in the following way

$$V_n = \min_{0=l_0 \le l_1 \le ... \le l_{n-1} \le l_n = 1} \sum_{j=0}^{N_j} \left| e^n\left(l_0^x + \Delta_{x1}\frac{j}{N_j}\right)\right|, \tag{9}$$

where V_n is the approximation error, $N_j >> n$ where N_j is the number of discrete evaluation points of function $h(x_1)$ along x_1 axis.

IV. Constrained Time-Optimal PWA Control

The control objective is to drive the system to the origin in minimum time. Let the sequence of control actions be written in the following way

$$\{u(k), u(k+1), \ldots, u(k+N-1)\}. \tag{10}$$

The optimal control sequence minimizes the number of time steps needed to reach a target region $\mathcal{X}_T \subseteq \mathcal{X}$ from the current state $\mathbf{x}(k)$. The constrained time-optimal control or constrained minimum-time control problem is defined in the following way:

$$\min_{u(k), u(k+1), \ldots, u(k+N-1)} N, \tag{11}$$

$$\begin{aligned} &\mathbf{x}(k+i) \in \mathcal{X},\ i = 0, \ldots, N-1, \\ &\mathbf{x}(k+N) \in \mathcal{X}_T, \\ &u(k+i) \in \mathcal{U},\ i = 0, \ldots, N-1, \end{aligned} \tag{12}$$

where *N* denotes the number of time steps needed to reach the target set and (1)–(2) are held for full control horizon, i.e. $k, k+1, ..., k+N-1$.

Feedback controllers for PWA systems can be computed using e.g. the Multi-Parametric Toolbox (MPT) for Matlab [4], [6]. The associated solution for different states can be stored into memory and then approximated by PWA state feedback law. Number of regions of the controller depends on the required controller accuracy. In general number of polyhedral partitions of the model *n* and the controller N_{reg} is different. Usually N_{reg} is larger than *n*. The LQR set calculated around the origin guarantee local stability of the closed control system [1]–[3], [10], [11].

V. Numerical Example – Car on the Nonlinear Hill

The system under consideration is the car on the nonlinear hill. Main aim of the control is to drive the car towards the origin. The model generally is non-linear due to trigonometric angle dependent functions. Nonlinear continuous-time model of the car can be written as follows

$$\ddot{z} + \frac{b_f}{m}\dot{z} + h(z) = \frac{1}{m}F, \tag{13}$$

where discrete-time model in state dependent form can be written by (1)–(2) with the following coefficients *a*, *b*:

$$\begin{cases} a = \dfrac{1}{m}, \\ b = \dfrac{b_f}{m}, \end{cases} \tag{14}$$

where m – mass of the car, b_f – coefficient of linear velocity dependent viscous friction, variable z in continuous time corresponds to variable x_1 in discrete-time. The nonlinear function $h(x_1)$ is defined by a nested trigonometric functions

$$h(x_1) = \sin\left(\text{arctg}\left(\quad (x_1)\right)\right), \tag{15}$$

where:

$$(x_1) = \frac{®}{10}\cos\left(\frac{®}{10}x_1 - \frac{3}{2}®\right)q(x_1), \tag{16}$$

$$q(x_1) = \begin{cases} 2, & -10 \le x_1 \le 10, \\ 1, & \text{otherwise}. \end{cases} \tag{17}$$

Functions $h(x_1)$, (x_1), $q(x_1)$ are constructed in order to model smooth nonlinear hills, where hill in the origin has two times higher altitude than surrounding hills.

Nonlinear function $h(x_1)$ with their three different secant approximation are plotted in Fig. 1. Solid line – function $h(x_1)$, dash-dotted line – their secant approximations with 3 partitions $h_a^3(x_1)$, dashed line – their secant approximations with 5 partitions $h_a^5(x_1)$ and dotted line – their secant approximations with 13 partitions $h_a^{13}(x_1)$.

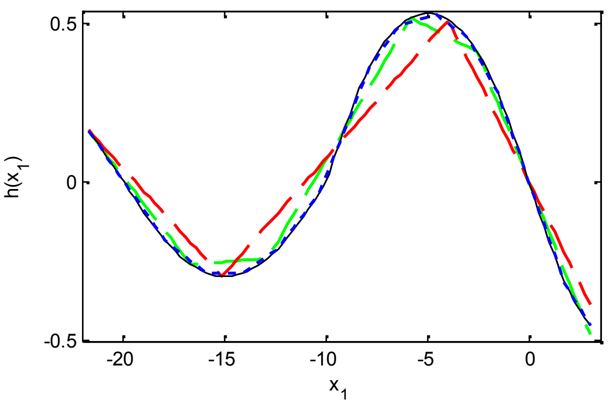

Fig. 1. Function $h(x_1)$ – solid line and their secant PWA approximations – $h_a^3(x_1)$ – dash-dotted line, $h_a^5(x_1)$ – dashed line, $h_a^{13}(x_1)$ – dotted line.

PWA approximation error of the nonlinear function $h(x_1)$ depends on the number of polyhedral partitions. Relationship between approximation error and number of PWA sectors in function $h_a^n(x_1)$ is depicted in Fig. 2.

The more partitions have the PWA model the more regions have the controller evaluated from (11). Relation between the number of controller regions and the number of PWA sectors in function $h_a^n(x_1)$ for the car on the nonlinear hill model is depicted in Fig. 3. Solution of the PWA controller has been computed using MPT Toolbox for Matlab [6]. The smallest number of controller regions is achieved for n = 3. Computation time in the considered example is strongly dependent with the number of controller regions, for $N_{reg} \cong 7.5 \cdot 10^4$ is about 9 hours.

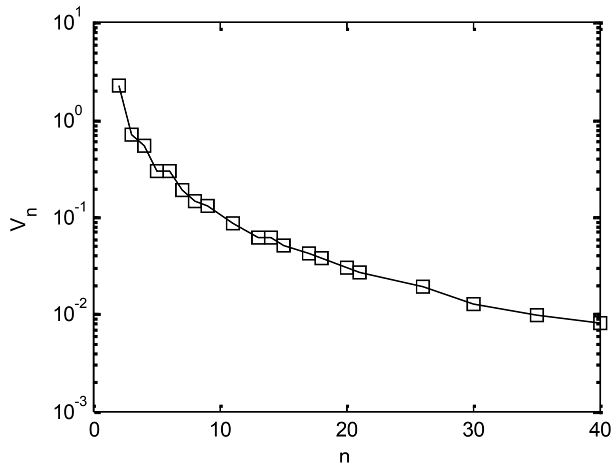

Fig. 2. Approximation error vs. number of PWA sectors in function $h_a^n(x_1)$.

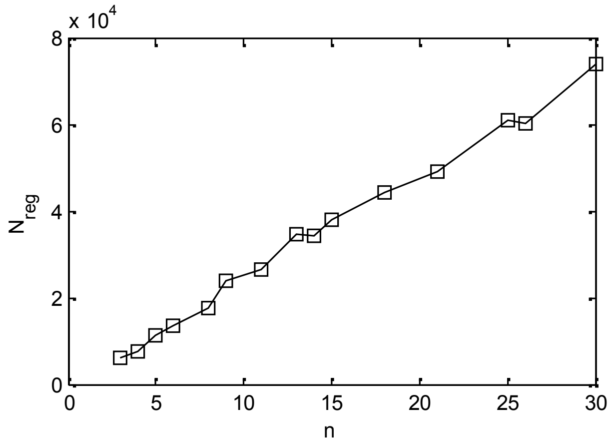

Fig. 3. Number of controller regions vs. number of PWA sectors in function $h_a^n(x_1)$.

State responses and control signal simulated for system approximated by $h_a^{13}(x_1)$ with initial conditions: $\mathbf{x}(0) = [-10, 0]^T$ are depicted in Fig. 4. System states are plotted by dashed lines: position of the car – state x_1 – thick one, velocity of the car – state x_2 – thin one. Solid line shows the control action during simulation horizon. On the basis of the controller computed for $h_a^{13}(x_1)$ it has been simulated responses for the reference system – the car with nonlinear function approximated by 2400 fixed length partitions $h_{fa}^{2400}(x_1) \approx h(x_1)$. States are plotted by dash-dotted lines: position x_1 – thick one, velocity x_2 – thin one. Control action for the reference system is plotted by dotted line. As it is shown in Fig. 5 the differences between both approximations $h_a^{13}(x_1)$, $h_{fa}^{2400}(x_1)$ are clearly visible, but not very significant.

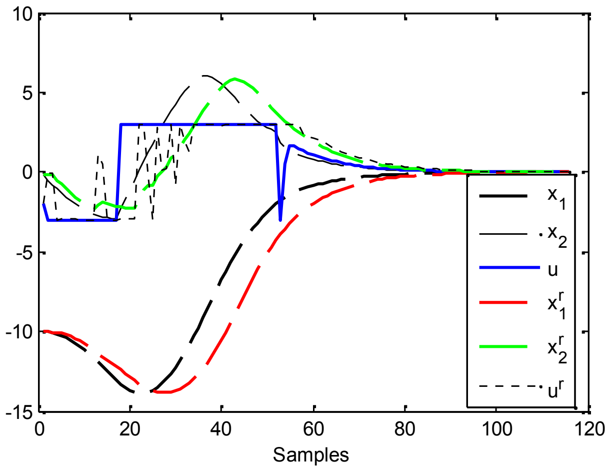

Fig. 4. System responses vs. time for initial conditions: $\mathbf{x}(0)=[-10,0]^T$ for the system approximated by $h_a^{13}(x_1)$: dashed line - states, solid line - input, and for the reference system $h_a^{2400}(x_1)$: dash-dotted lines – states, dotted line - input.

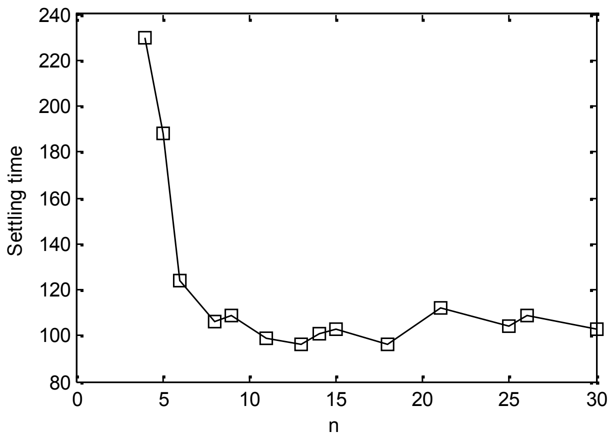

Fig. 5. Settling time of the output response of the system with initial conditions: $\mathbf{x}(0)=[-10,0]^T$ and controller computed using PWA model with *n* sectors in function $h_a^n(x_1)$ for the reference plant $h_{fa}^{2400}(x_1)$.

Performance of the closed loop control system is evaluated in close to real conditions. PWA controller computed using approximated PWA model with *n* partitions operates the reference plant. The reference plant has 2400 fixed length partitions, which is practically equivalent to the nonlinear plant. The performance analysis is made on the basis of settling time of the control system. The settling time is computed for each controller and depicted in Fig. 5. For *n* = 3 the feedback control system is unstable. As it can be seen from Fig. 5 settling time decreases very fast for $n \in \langle 3,8 \rangle$, and then for $n \geq 8$ the settling time is approximately equal to 100 seconds.

VI. CONCLUSIONS

Performance of the nonlinear control system with off-line PWA controller tuned for approximated PWA model has been analysed. On the basis of the results shown in Fig. 1 and Fig. 2 it may be concluded that for the analysed secant approximation of the nonlinear function, accuracy grows when the number of the polyhedral partitions in the function $h(x_1)$ increases. The closed loop performance of the system is evaluated using the settling time, which is understood as number of time steps needed to reach a target region $\mathcal{X}_T$. It is shown in Fig. 5 that the settling time decreases with the number of polyhedral partitions only until some certain number of sectors e.g. until $n \in [8,13]$.

As shown in Fig. 5 the performance of the system does not depend substantially on the number of partitions in the PWA model for n > 13. Nevertheless Fig. 3 clearly shows that complexity of the controller grows with increasing number of partitions. Comparison of Fig. 3 and Fig. 5 shows that, an optimal number of PWA model partitions can be found due to computational complexity and performance of the system.

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