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Rational Polynomial Windows as an Alternative for Kaiser Window

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Introduction

Time windows applied in signal processing applications are typically used for the reduction of the spectral leakage effect caused by the finite number of discrete samples representing the analyzed signal. The spectral leakage is in fact the result of the rapid cutting of the signal which is equivalent to the application of the rectangular window. In order to reduce this phenomenon some other window functions have been proposed e.g. simple triangular window or famous raised-cosine family represented by well-known Hann, Hamming or Blackman windows [1, 2].

Some other applications of the window functions are related to the reduction of the Gibbs phenomenon in the frequency domain caused by the finite order of trigonometric approximation of periodic functions as well as some applications related to short-time Fourier transform (STFT) and the design of Finite Impulse Response (FIR) filters.

In many DSP applications where the number of analyzed samples is constant, there is a possibility of storing the pre-computed values of the windowing function but some adaptive signal analysis algorithms often require the usage of the windows with variable length. In such case the computational complexity of window function plays an important role, especially if the memory amount is limited.

Window functions and their parameters

In order to compare the window functions some parameters are typically used which allow the proper choice of the window according to specific application's requirements. The most relevant of them may be calculated in the frequency domain, since they are related to the properties of the window's spectrum e.g.:

- Width of the Main Lobe (WML);
- Highest Sidelobes Level (HSLL);
- Asymptotic decay ratio (in dB per octave);
- Energetic criterion.

The comparison of the frequency properties of different windows is usually conducted assuming the same WML, since its increase causes the "automatic" improvement of the HSLL or the energetic criterion. Since the most popular windows (e.g. Hamming or Hann) have the main lobe twice as wide as the rectangular window (WML=2), the optimization of the parameters of newly developed windows can be conducted for the same WML.

Optimizing some of the parameters mentioned above, some well-known windows have been obtained [1] e.g. Dolph-Chebyshev window minimizing the level of the sidelobes (HSLL) with 0 dB/octave decay ratio or Kaiser window nearly optimal in the energetic sense.

Recently, some other window functions have been proposed by various researchers e.g. cosine hyperbolic windows [3], application of the sinc function [4] and even the modified version of classical Hamming window [5].

Some of them, similarly as Kaiser window, can be controlled by some specific parameters allowing the choice of a compromise between the optimal properties e.g. the main lobe and sidelobes' peak as for unispherical windows [6]. This family of windows is based on Gegenbauer polynomials [7], being the generalization of Legendre polynomials and the special cases of Jacobi polynomials.

Nevertheless, the typical disadvantage of those windows is their high computational complexity limiting the potential areas of effective applications, especially for adaptive analysis. A typical example of this problem is Kaiser window where Bessel function should be expanded into high order polynomial in order to achieve the accuracy of calculations required to obtain good properties of the window's characteristics.

Computational complexity and energetic criterion

Expanding the cosine functions or Bessel function into N-th order polynomial, the computational complexity of $2 \cdot N$ arithmetic operations for each point is obtained, according to Horner's scheme. Since Kaiser window for the observation time T is defined as

$$wK(t) = \frac{I_0\left(\beta \cdot \sqrt{1-\left(\frac{t}{T}\right)^2}\right)}{I_0(\beta)} \cdot \prod\left(\frac{t}{2 \cdot T}\right), \qquad (1)$$

where $I_0(x)$ denotes the modified Bessel function of the first kind and zero order, in most practical applications this function can be expanded into polynomial series. The recommended polynomial is at least 10-th order one and the decrease of its order causes the worsening of the window parameters, especially for low order polynomials.

Assuming the parameter $\beta = \pi \cdot \sqrt{3}$ and T=1 the WML=2 is obtained, which is the same as for Hann and Hamming windows.

Kaiser window is in fact the approximation of the spheroidal window [8], which is optimal in the energetic sense i.e. minimizes the energetic criterion defined as the ratio of the side lobes' to the main lobe's energy. It can be calculated for each window function using the main lobe's and the total energy

$$\alpha = \frac{E_{SL}}{E_{ML}} = \frac{E_{total} - E_{ML}}{E_{ML}} = \frac{E_{total}}{E_{ML}} - 1, \qquad (2)$$

where the total (common) energy can be computed in the frequency domain as

$$E_{total} = \int_{-\infty}^{\infty} [U(f)]^2 df \qquad (3)$$

or, according to Parseval's theorem for symmetrical window function, in the time domain as

$$E_{total} = \int_{T}^{T} [w(t)]^2 dt = 2 \cdot \int_{0}^{T} [w(t)]^2 dt. \qquad (4)$$

Calculation of the main lobe's energy can be conducted according to the formula

$$E_{ML} = 2 \cdot \int_{0}^{WML} [U(f)]^2 df, \qquad (5)$$

where $U(f)$ is the window's spectrum in the frequency domain and $w(t)$ is the time domain window function. The spectrum can be computed using the Fourier transform as

$$U(f) = \int_{-T}^{T} w(t) \cdot \cos(2 \cdot \pi \cdot f \cdot t) dt. \qquad (6)$$

Obtained spectral amplitude characteristics are typically presented in the logarithmic form as $20 \cdot \log(|U(f)|)$.

Table 1. Energetic criterion for various order of polynomials used as the approximation of Bessel functions for Kaiser window

Order of polynomial	Energetic criterion α
5	$1.31425 \cdot 10^{-4}$
6	$1.21310 \cdot 10^{-4}$
7	$1.19715 \cdot 10^{-4}$
8	$1.19518 \cdot 10^{-4}$
9	$1.19499 \cdot 10^{-4}$
10	$1.19497 \cdot 10^{-4}$

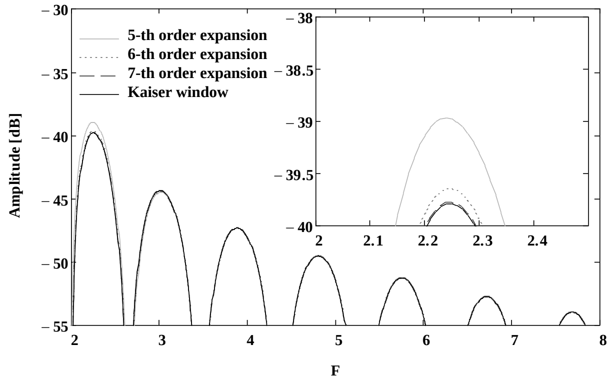

Fig. 1. Comparison of the first side lobes of the frequency characteristics obtained for various order of polynomials used as the approximation of Bessel functions for Kaiser window

The energetic criterion for Kaiser window with WML=2 is equal to α=$1.19497 \cdot 10^{-4}$ and its values obtained by the expansion of the Bessel functions into polynomial series for Kaiser window are presented in Table 1.

The comparison of the side lobes of such obtained windows and Kaiser window is illustrated in Fig. 1.

The idea of polynomial windows

Since many window functions, such as raised-cosine family or Kaiser window in practical implementations are approximated by polynomials, there is a possibility to use the polynomials directly as the window functions and optimize them according to various criterions. The general definition of such polynomial windows is given as

$$w_p(t) = 1 + \sum_{n=1}^{N} \left[C_n \cdot \left| \frac{t}{T} \right|^n \right], \qquad (6)$$

where N is the order of polynomial. Nevertheless, much better properties of such windows can be obtained using only even exponents [9, 10], what leads to the following definition

$$w_p(t) = 1 + \sum_{n=1}^{N/2} \left[C_{2n} \cdot \left(\frac{t}{T} \right)^{2n} \right]. \qquad (7)$$

In that case the symmetry of the window function of N-th order in the time domain is obtained directly without the use of the absolute value operators. Assuming T=1 and applying the substitution x=t^2 the obtained computational complexity using Horner's scheme is equal to $N/2$+1 operations (N/4 multiplications, N/4 summations and the additional substitution) for $N/2$-th order window (and N+1 for the N-th order window).

The spectrum of the window (7) can be calculated using Fourier transform as

$$U_p(t) = \frac{\sin(\pi \cdot f \cdot T)}{\pi \cdot f} + \sum_{n=1}^{N/2} (C_{2n} \cdot I_{2n}), \qquad (8)$$

where the recurrent definition of I_{2n} is given as [10]

$$I_{2n} = \frac{T \cdot [(\pi \cdot f \cdot T) \cdot \sin(\pi \cdot f \cdot T) + 2 \cdot n \cdot \cos(\pi \cdot f \cdot T)]}{4^n \cdot (\pi \cdot f \cdot T)^2} - $$
$$- \frac{n \cdot (2 \cdot n - 1)}{2 \cdot (\pi \cdot f \cdot T)^2} \times I_{2n-2} \quad (9)$$

and then normalised dividing by

$$U(0) = 1 + \sum_{n=1}^{N/2} \frac{C_{2n}}{4^n \cdot (2 \cdot n + 1)}. \quad (10)$$

In order to obtained the desired properties of the window the optimisation of the polynomial coefficients with additional constraints (e.g. related to the specified WML) can be performed leading e.g. to the family of polynomial windows with low side lobes' level proposed in one of the earlier papers [10].

It is worth to notice that the comparison of the frequency properties of various windows should be conducted for the same WML. Comparing e.g. the side lobes' level the same asymptotic decay ratio should also be assured what can be obtained forcing the specified number of derivatives to zero at both ends of the window ($t=\pm T/2$). Using the polynomial windows obtained relations are relatively simple because of the simple form of each derivative.

Due to their low computational complexity polynomial windows can be more useful for adaptive windowing than e.g. cosine-based ones. Polynomial functions can also be applied in some other signal technology and computer science applications e.g. for efficient ECG modelling [11] or curve fitting.

Proposed family of rational polynomial windows and their energetic optimization

Polynomial functions, especially for low order of polynomials not always allow obtaining desired properties. Considering some other applications of polynomials e.g. B-spline curve modelling used in computer graphics and visualisation technology, it can be stated that there are some possibilities of modifications dependent on the specific application. In order to obtain more universal definitions of smooth curves for computer graphics the idea of Non-uniform Rational B-Spline (NURBS) functions has been proposed. A similar idea is known as Padé approximant as an alternative for the function's expansion into Taylor series.

The application of such ideas for the polynomial windows leads to the definition of rational polynomial window expressed as

$$w_{R(N,M)}(t) = \frac{1 + \sum_{n=1}^{N/2} \left[a_{2n} \cdot \left(\frac{t}{T} \right)^{2n} \right]}{1 + \sum_{m=1}^{M/2} \left[b_{2m} \cdot \left(\frac{t}{T} \right)^{2m} \right]} \quad (11)$$

where $N/2$ and $M/2$ are the polynomial orders for the numerator and denominator respectively and only even exponents are present in both polynomials. The overall computational complexity of such window is equal to $N+M+2$ operations (N and M arithmetic operations for both polynomials, 1 for the substitution and one for the division of both polynomials). In such sense the application of such defined rational polynomial window instead of the polynomial window (7) leads to lower computational complexity if $N+M$ is less than the order of polynomial window (7) or the half of the order of polynomial used for the expansion into series (the computational complexity of the "full" polynomial of N-th order, with all exponents, is equal to $2 \cdot N$ operations). Nevertheless, using the rational functions a better fitting of window function to the "ideal" shape of the window is possible.

The energetic optimisation of the rational polynomial window has been conducted assuming the WML=2 in order to compare the obtained results with those presented in Table 1. As the basic function the ration of two 2-nd order polynomials has been chosen. In such case only the one polynomial coefficient (e.g. a_2) can be optimized, since the second one (b_2) is calculated numerically in order to guarantee the WML=2. The other calculations have been performed for higher order of polynomials, but still one of the coefficients has been calculated using the constraint related to WML=2.

The values of the energetic criterion obtained for the proposed family of low order rational polynomial windows are presented in Table 2 and the first side lobes of their frequency characteristics are illustrated in Fig. 2.

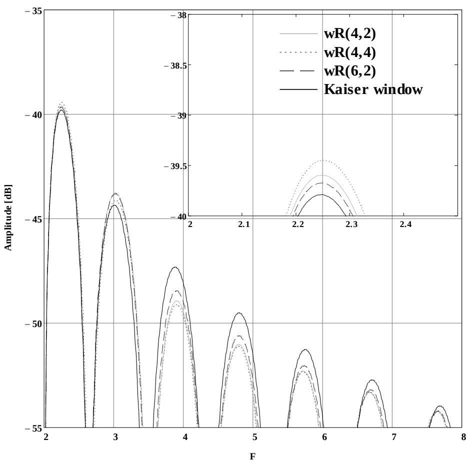

Fig. 2. Comparison of the first side lobes of the frequency characteristics obtained for proposed family of windows and Kaiser window

Table 2. Energetic criterion for various orders of polynomials in the proposed windows family

window	N	M	Energetic criterion α	Computational complexity
$w_{R(2,2)}$	2	2	$3.77597 \cdot 10^{-4}$	6 operations
$w_{R(4,2)}$	4	2	$1.20955 \cdot 10^{-4}$	8 operations
$w_{R(4,4)}$	4	4	$1.20836 \cdot 10^{-4}$	10 operations
$w_{R(6,2)}$	6	2	$1.20312 \cdot 10^{-4}$	10 operations

Table 3. Polynomial coefficients of the obtained rational polynomial windows

window	a_2	a_4	a_6	b_2	b_4
$w_{R(2,2)}$	-1	---	---	1.805	---
$w_{R(4,2)}$	-1.673	0.723	---	0.820	---
$w_{R(4,4)}$	-1.673	0.724	---	0.813	0.033
$w_{R(6,2)}$	-1.753	0.885	-0.086	0.733	---

Conclusions

Since the expansion of the Bessel functions into 6-th order series (computational complexity of 12 operations) leads to the energetic criterion equal to $1.21310 \cdot 10^{-4}$ even simple $w_{R(4,2)}$ window leads to better energetic properties, demonstrating the usefulness of the proposed approach.

The frequency characteristics of obtained window (except $w_{R(2,2)}$) are very similar to Kaiser window and the HSLL is still below 39.5 dB similarly as for the 6-th order expansion of the Bessel function.

Proposed rational polynomial windows can also be optimized using various other criterions e.g. minimizing the HSLL for a specified asymptotic decay ratio, leading to good properties preventing low computational complexity.

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K. Okarma. Rational Polynomial Windows as an Alternative for Kaiser Window // Electronics and Electrical Engineering. – Kaunas: Technologija, 2012. – No. 7(123). – P. 91–94.

In the paper a new family of energetically optimized rational polynomial windows useful for signal processing applications is presented. A typical approximation of the energetically optimal spheroidal window is well-known Kaiser window which can be calculated using the Bessel function. In practical implementations this function should be expanded into polynomial series of a specified order affecting the values of the energetic criterion being the ratio of the side lobes' energy to the energy of the main lobe of the window's spectrum. The extension of the previously proposed polynomial windows family into rational polynomial windows presented in this paper leads to good approximation of Kaiser window with seriously reduced computational complexity in comparison to the expansion into the polynomial series. The window's value for each sample can be efficiently computed using Horner's scheme reducing the number of arithmetic operations. Ill. 2, bibl. 11, tabl. 3 (in English; abstracts in English and Lithuanian).

K. Okarma. Racionalūs polinominiai langai kaip Kaiserio lango alternatyva // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2012. – Nr. 7(123). – P. 91–94.

Pateikiama naujų energetiškai optimizuotų racionalaus polinomo langų, naudojamų signalų apdorojimo sistemose šeima. Tipinė energetiškai optimalaus sferoidinio lango aproksimacija yra gerai žinomas Kaiserio langas, kuris gali būti skaičiuojamas naudojant Beselio funkciją. Praktikoje ši funkcija turi būti išplėsta į tam tikro laipsnio polinomus. Polinomo langų šeimos išplėtimas į racionalaus polinomo langus leidžia gauti gerą Kaiserio lango aproksimaciją gerokai sumažinant skaičiavimo kompleksiškumą. Kiekvienos imties lango vertė gali būti efektyviai apskaičiuota naudojant Hornerio schemą, taip sumažinant aritmetinių operacijų skaičių. Il. 2, bibl. 11, lent. 3 (anglų kalba; santraukos anglų ir lietuvių k.).