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Generalised Fractional Indexes Approximation with Application to Discrete-Time Generalised Weyl Symbol Computation

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Introduction

Dynamical, linear discrete-time system can be described by finite set of coefficients of difference equations or state space model. The set define dynamics of the linear time-invariant system for all times $k \in \mathbb{Z}$ (infinite time horizon). In contradistinction the description of discrete-time linear time-varying systems requires in general definition of an infinite number of coefficients.

In order to describe dynamics of time-varying discrete-time systems one can use following state space description equations with time-dependent matrices [1, 2, 16, 18]:

$$\mathbf{x}(k+1)=\mathbf{A}(k)\mathbf{x}(k)+\mathbf{B}(k)\mathbf{v}(k), \qquad (1)$$

$$\mathbf{y}(k)=\mathbf{C}(k)\mathbf{x}(k)+\mathbf{D}(k)\mathbf{v}(k), \qquad (2)$$

where $k \in \mathbf{Z}$, $\mathbf{x}(k) \in \mathbf{R}^{n_k}$, $\mathbf{v}(k) \in \mathbf{R}^{m}$, $\mathbf{y}(k) \in \mathbf{R}^{p}$, $\mathbf{A}(k) \in \mathbf{R}^{n_k \times n_k}$, $\mathbf{B}(k) \in \mathbf{R}^{n_k \times m}$, $\mathbf{C}(k) \in \mathbf{R}^{p \times n_k}$, $\mathbf{D}(k) \in \mathbf{R}^{p \times m}$ and $\mathbf{n}=\left\{n_k : k \in \mathbf{Z}, n_k \in \mathbf{N}, \mathbf{x}(k) \in \mathbf{R}^{n_k}\right\}$.

Above model can be converted into more general operators description [1, 2, 16, 18] with transfer operator defined by set of impulse responses

$$\hat{\mathbf{T}}=\begin{bmatrix} h_{0,0} & 0 & \cdots & 0 & 0 \\ h_{1,0} & h_{1,1} & \cdots & \vdots & \vdots \\ h_{2,0} & h_{2,1} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & h_{N-2,N-2} & 0 \\ h_{N-1,0} & \cdots & \cdots & h_{N-1,N-2} & h_{N-1,N-1} \end{bmatrix}, \qquad (3)$$

where h_{k_1,k_0} denotes system (1)-(2) response to Kronecker delta $\delta(k-k_0)$ at time k_1, i.e. after $k_1 - k_0$ samples.

Nevertheless analyzing or processing data with infinite dimensional size is impossible. Additional simplifying assumptions allow one to describe the time-varying system with finite set of coefficients. Linear time-varying systems can be classified with respect to the simplifying assumption. Generally following classes of time-varying systems can be distinguished [3]: general time-varying, periodic time-varying, almost periodic time-varying, almost time-invariant. Independently on the class of the system, but especially for time-varying systems in the general form analysis can be realized only on finite time horizon. It mean that accessible system data is limited by two constraints for indexes $k_{\min}$ and $k_{\max}$ that define range for variable k

$$k \in \left\{k{:}k{\in}\mathbf{Z}\backslash\left\{k{<}k_{\min} \vee k{>}k_{\max}\right\}\right\}. \qquad (4)$$

There are no assumptions about past and future system behaviour.

Time-frequency methods for continuous time systems are well known [4–10] as well as frequency methods for discrete-time systems [11–14]. Many investigations has been made until now. Recently there are also known successful applications of time-varying approximations for nonlinear systems [17]. The time-frequency transform is formulated as parameterized extension of Laplace transform. General form of the transform for continuous time systems can be defined by generalised Weyl symbol [10, 15].

Discrete-time formula of the Generalised Weyl Symbol can be written using digital set of parameterised impulse responses (5)

$$h_{k,n}^{(\alpha)}=h\left(k+\left(\tfrac{1}{2}-\alpha\right)n, k-\left(\tfrac{1}{2}+\alpha\right)n\right) \qquad (5)$$

and the Discrete Fourier Transform (DFT) in following way

$$L_{k,l}^{(\alpha)}=\sum_{n=1}^{N} h_{k,n}^{(\alpha)} e^{-j2\pi ln/N}, \qquad (6)$$

where $\alpha \in \mathbf{R}$ is arbitrary real number, usually bounded such that $|\alpha| \le 0.5$ and is system response at time k_1 for shifted by time k_0 Kronecker delta δ_{k,k_0}, variable $k = 0,1,\ldots,N$ represents discrete time $t_k = kT_p$ and $l = 0,1,\ldots,N/2-1$ is connected with frequency, where discrete frequency $\Omega_l = l/N$ and analogue frequency $f_l = l/NT_p$ where T_p is sampling period for digitalized systems.

Time-frequency transformation can be computed directly from eq. (6) only for $\alpha = \pm 0.5$ (time-varying Zadeh transfer function [4] $\alpha = 0.5$ and frequency dependent modulation function [5] known also as Kohn-Nirenberg symbol [9] $\alpha = -0.5$). For $\alpha = 0$ one can apply fractional indexes approximation introduced in [15].

Main aim of the paper is to develop new generalised fractional indexes – computational method which allows to determine generalised Weyl symbol for arbitrary real $\alpha = \langle -0.5, 0.5 \rangle$, not only for $\alpha = \pm 0.5$ (integer indexes method) and $\alpha = 0$ (fractional indexes [15]). Parameter α allows to shape the set of parameterised impulse responses. The selection of the parameter α in the generalised Weyl symbol enables selection of the best accuracy region for the time-frequency transform.

Generalised fractional indexes approximation

Definition. Generalised fractional index discrete time response value of one variable $h(p+\gamma),\ \gamma \in [0,1),\ p \in \mathbb{Z}$ is defined as linear interpolation of h taken in following way

$$h(p+\gamma) = (1-\gamma)h(p) + \gamma h(p+1). \tag{7}$$

Definition. Generalised fractional indexes discrete time response value of two variables $h(p+\gamma, m+\beta),\ \gamma,\beta \in [0,1),\ p,m \in \mathbb{Z}$ is defined as 2-D linear interpolation of h taken in following way

$$\begin{aligned} &h(p+\gamma, m+\beta) = \\ &= (1-\gamma)(1-\beta)h(p,m) + \gamma(1-\beta)h(p+1,m) + \\ &+ (1-\gamma)\beta h(p,m+1) + \gamma\beta h(p+1,m+1). \end{aligned} \tag{8}$$

Taking account (5) we have:

$$\begin{cases} p_\gamma = k + \left(\tfrac{1}{2} - \alpha\right)n,\ p = \operatorname{floor}(p_\gamma),\ \gamma = p_\gamma - p, \\ m_\beta = k + \left(\tfrac{1}{2} + \alpha\right)n,\ m = \operatorname{floor}(m_\beta),\ \beta = p_\beta - p, \end{cases} \tag{9}$$

where floor denotes round toward minus infinity.

Generalized fractional impulse response can be written as follows

$$\begin{aligned} h_{k,n}^{(\alpha)} &= (1-\gamma)(1-\beta)h(p,m) + \gamma(1-\beta)h(p+1,m) + \\ &+ (1-\gamma)\beta h(p,m+1) + \gamma\beta h(p+1,m+1). \end{aligned} \tag{10}$$

Thus generalised discrete-time Weyl symbol approximation can be defined by substituting (10) in (6), in the following form:

$$L_{k,l}^{(\alpha)} = \sum_{n=1}^{N} \begin{pmatrix} (1-\gamma)(1-\beta)h(p,m) \\ +\gamma(1-\beta)h(p+1,m) \\ +(1-\gamma)\beta h(p,m+1) \\ +\gamma\beta h(p+1,m+1) \end{pmatrix} e^{-j2\pi ln/N}, \tag{11}$$

where variables $\underline{a}, \overline{a}, \underline{b}, \overline{b}, \gamma, \beta$ are defined above (9).

Application of the generalised fractional indexes for generalised Weyl symbol computation.

Time-invariant systems are always defined on infinite time horizon, thus all elements of the impulse response are always definite. Responses for time-varying systems do not need to be definite in general for all $k \in \mathbb{Z}$. The system is defined only on some bounded time horizon (4).

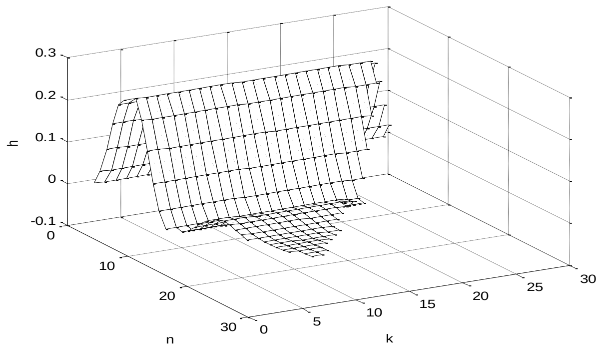

Fig. 1. Parameterized impulse response with $\alpha = -0.3$ and fractional indexes approximation for 4th order Butterworth filter defined on finite time horizon $k = 0,1,2,\ldots,29$

Computation of time-frequency Weyl symbol for systems defined on finite time horizon is not an easy task. Let us consider for example low-pass filter FIR Butterworth 4th order filter with cut-off frequency $\Omega_c = 0.2$. The time horizon N=30 and is bounded by (4) where $k_{\min} = 0$ and $k_{\max} = N-1$.

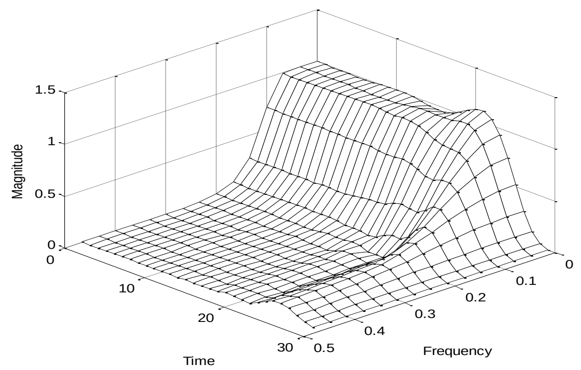

Fig. 2. 3D Magnitude-Time-Frequency diagram calculated for $\alpha = -0.5$ using fractional indexes impulse responses for system defined on finite time horizon

Impulse responses h_{k_1,k_2} available for computation must fit the time horizon $\underset{i=1,2}{\forall} k_{\min} \le k_i \le k_{\max}$. For at least one index outside the time horizon $\underset{i=1,2}{\exists} k_i < k_{\min}$ or $k_i > k_{\min}$ responses are indefinite. Fig. 1 shows impulse responses (5) accessible for computations in respect to discrete variables k and n and parameter $\alpha = -0.3$ in (5). Inaccessible responses for indexes outside the range (past and future) are not plotted.

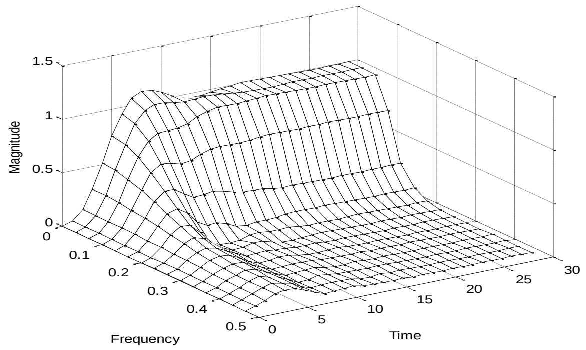

Fig. 3. 3D Magnitude-Time-Frequency diagram calculated for $\alpha = 0.5$ using fractional indexes impulse responses for system defined on finite time horizon

Applying parameterized impulse response (5) for the discrete-time low-pass filter mentioned above defined on finite horizon, three following 3D time-frequency diagrams are calculated and plotted in Fig. 2, Fig. 3 (integer indexes $\alpha = \pm 0.5$) and 4 (generalised fractional indexes $\alpha = -0.3$). Inaccessible impulse responses

$$\left\{ h(k_1, k_2) : \underset{i=1,2}{\exists} k_i < k_{\min} \text{ or } k_i > k_{\min} \right\}$$

are substituted for computations by zeros. In Fig. 2–Fig. 4 visible are finite time horizon boundary effects.

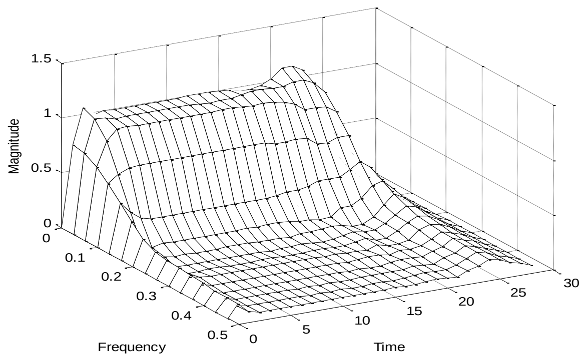

Fig. 4. 3D Magnitude-Time-Frequency diagram calculated for $\alpha = -0.3$ using fractional indexes impulse responses for system defined on finite time horizon

Fig. 2 shows Kohn-Nirenberg symbol [9] with $\alpha = -0.5$ and the high accuracy at the beginning part of the time horizon while in fig. 3 is plotted time-varying transfer function [4] with $\alpha = 0.5$ and the high accuracy at the end part of the time horizon. Fig. 4 is 3D magnitude of Generalised Weyl Symbol with $\alpha = 0$ calculated using fractional indexes approximation for impulse responses. The high accuracy are in the beginning-middle part of the time horizon. Accuracy for the end and beginning part of the time horizon is worse. Applying for computations generalised Weyl symbol with fractional indexes it is possible to choose precisely the part of the time horizon to compute with the high accuracy.

Conclusion

Time-frequency transformation is well known tool for systems and signals analysis. Accuracy of discrete-time, time-frequency diagrams depends mostly on the length of the time window. For systems defined on finite time horizon the length samples outside the time horizon are inaccessible. Generally there are two ways to analyse the system: use very short time-window, at least 2 times shorter then time horizon, or analyse the system on the full time horizon with incomplete data (without data outside the defined time horizon.

Short time horizons results in boundary effects (boundary distortions) on the time-frequency diagram – the beginning and the end of time horizon. Using additional parameter α one can continuously choose the best accuracy region from the finite time horizon. Negative values of the transformation parameter close to $\alpha = -0.5$ results in the best accuracy at the beginning of the time horizon, whereas positive values close to $\alpha = 0.5$ gives the best accuracy at the end of the time horizon. Middle values of α close to zero ensures the best accuracy in the middle of the time horizon.

References

1. **Dewilde P., Van der Veen A. J.** Time–Varying Systems and Computations. – Boston: Kluwer Academic Publishers, 1998.
2. **Halanay A., Ionescu V.** Time–Varying Discrete Linear Systems Input–Output Operators. Riccati Equations, Disturbance Attenuation // Operator Theory: Advances and App. – Basel–Boston–Berlin: Birkhauser, 1994. – Vol. 68.
3. **D'Angelo H.** Linear Time–Varying Systems. – Boston: Allyn and Bacon, 1970.
4. **Zadeh L. A.**: Frequency analysis of variable networks // Proc. of the Institute of Radio Engineers, 1950. – Vol. 38. – P. 291–299.
5. **Bello P. A.** Characterisation of randomly time–variant linear channel // IEEE Tran Comm. Syst., 1963. – Vol. 11. – P. 360–393.
6. **Hlawatsch F.**: Time–frequency analysis and synthesis of linear signal spaces: Time–frequency filters, signal detection and estimation, and range–Doppler estimation. – Boston (MA): Kluwer, 1998.
7. **Hlawatsch F., Matz G.** Nonstationary Spectral Analysis Based on Time–Frequency Operator Symbols and Underspread Approximations // IEEE Transactions on Information Theory, 2006. – Vol. 52(3). – P. 1067–1086.
8. **Hlawatsch F., Matz G.** Wigner distributions (nearly) everywhere: time–frequency analysis of signals, systems, random processes, signal spaces, and frazes // Signal Processing, 2003. – Vol. 83. – P. 1355–1378.
9. **Kohn J. J., Nirenberg L.** An algebra of pseudo–differential operators // Comm. Pure Appl. Math., 1965. – Vol. 18. – P. 269–305.
10. **Kozek W.** On the generalized transfer function calculus for underspread LTV channels // IEEE Trans. Signal Processing, 1997. – Vol. 45. – P. 219–223.

11. **Orlowski P.** Frequency Domain Analysis of Uncertain Time–Varying Discrete–Time Systems // Circuits, Systems and Signal Processing, 2007. – No. 3(26). – P. 293–310.
12. **Orlowski P.** Comparison of frequency domain methods for discrete–time, linear time–varying system with invariant eigenvalues // International Journal of Factory Automation, Robotics and Soft Computing, 2007. – No. 4. – P. 27–32.
13. **Orlowski P.** System Degradation Factor for Networked Control Systems // Information Technology and Control, 2008 – No. 3(37). – P. 233–244.
14. **Orlowski P.** Simplified Design of Low–Pass, Linear Parameter–Varying, Finite Impulse Response Filters // Information Technology and Control, 2010. – No. 2(39). – P. 130–137.
15. **Orlowski P.** Fractional indexes impulse responses approximation for discrete–time Weyl Symbol computation // Electronics and Electrical Engineering – Kaunas: Technologija, 2010. – No. 8(104). – P. 9–12.
16. **Orlowski P.** Discrete–Time, Linear Periodic Time–Varying System Norm Estimation Using Finite Time Horizon Transfer Operators // Automatika. – Korema, 2010. – No. 4(51). – P. 325–332.
17. **Orlowski P.** Convergence of the Discrete–Time Nonlinear Model Predictive Control with Successive Time–Varying Linearization along Predicted Trajectories // Electronics and Electrical Engineering – Kaunas: Technologija, 2011. – No. 7(113). – P. 27–31.
18. **Orlowski P.** Generalized feedback stability for periodic linear time–varying, discrete–time systems // Bulletin of the Polish Academy of Sciences: Technical Sciences – Polish Academy of Sciences, 2012. – No. 1(60). – P. 171–178.

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Unique description of discrete-time, linear time-invariant systems on infinite time horizon requires only definition of finite number of coefficients, usually relatively small. In contradistinction the description of discrete-time linear time-varying systems requires in general definition of an infinite number of coefficients. Nevertheless neither analyzing nor processing data with infinite dimensional size is impossible. The main aim of the paper is to develop new generalised fractional indexes – computational method which allows to determine generalised Weyl symbol for arbitrary real α not only for $\alpha=\pm0.5$ (integer indexes) and $\alpha=0$. Parameter α allow to shape the set of parameterised impulse responses. The selection of the parameter α in the generalised Weyl symbol enable selection of the best accuracy region for the time-frequency transform. Numerical examples illustrates how the approximation of the system response with generalised fractional indexes increase accuracy for the computation of the discrete-time, time-frequency transformation calculated on finite time horizon. Ill. 4, bibl. 18 (in English; abstracts in English and Lithuanian).

P. Orlowski. Apibendrinta trupmeninių indeksų aproksimacija naudojant apibendrintų Veilo simbolių skaičiavimą // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2012. – Nr. 7(123). – P. 13–16.

Skirtingai nuo diskrečiųjų laiko sistemų, tiesinėms laikui bėgant kintančioms sistemoms aprašyti reikia apibrėžti begalinį koeficientų skaičių. Tačiau begalinio dydžio duomenų nei analizuoti, nei apdoroti neįmanoma. Pagrindinis šio darbo tikslas sukurti naują apibendrintų trupmeninių indeksų skaičiavimo metodą, kuris leistų nustatyti apibendrintą Veilo simbolį realiam kintamajam α, ne tik kai $\alpha=\pm0{,}5$ (sveikųjų skaičių indeksai), bet ir kai $\alpha=0$. Parametras α leidžia apibrėžti impulse parametrizuotų atsakų rinkinį. Skaičiavimų pavyzdžiai rodo, kaip sistemos atsako aproksimavimas, naudojant apibendrintus trupmeninius indeksus, padidina skaičiavimo tikslumą. Il. 4, bibl. 18 (anglų kalba; santraukos anglų ir lietuvių k.).