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A Method for Exact Error Probability Determination of Nonuniform Signaling For Gaussian Channels

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Introduction

In the conventional approach to data transmission , each point in a given constellation is equally likely to be transmitted. Although this approach gives the maximal bit rate for a given constellation size, it does not take into account the energy cost of the various constellation points. The idea of choosing constellation points with a nonequiprobable distribution was explored in [1]. The nonuniform signaling is more general for consideration and has better performance than the equiprobable transmission. The optimal nonuniform signaling was considered in [1], but the error probability was evaluated for great signal-to-noise ratios (SNR) values and medium numbers of the nearest neighbours.

The performance of specific signal constellations in digital communications problems is often described through use of the union bound, the minimum distance bound and the closest neighboring method [2]. A new upper bound was presented in B. Hughes' paper [3]. This bound can be applied to any digital set and it is always better than union bound and minimum distance bound. An asymptotically tight lower bound was considered in P. F. Swaszek's paper [4].

The calculations in previous mentioned papers are based on the equiprobable transmission and there are more or little close to the exact error probability.

A simple method for exact error probability determination of nonuniform signaling in two dimensions (2D) for Gaussian channels is presented in this paper. The proposed method will be illustrated on the example of iterative polar quantization [5] and comparison among this method and some of previous mentioned methods will be done. The exact error probability determination make possible the exact nonuniform signal constellation analysis for any SNR value. The proposed method is simpler than approximate methods presented in [3] and [4], and it is more easy for implementation in 2D space.

The aim of this paper is not in signal constellation optimization, but in approach to exact error probability determination. Therefore, the analysis is done for quantizations obtained using the polar quantizations that minimize the mean-squared error. These quantizations are useful for implementation on circle symmetrical sources (for example Gaussian source).

A Method for exact error probability determination of 2D signal constellations

Consider the problem of detecting one of L nonequiprobable signals in additive white Gaussian noise. For a 2D signal constellation representation , the observed date are

$$\boldsymbol{r}=O_{ij}+\boldsymbol{n},$$

where O_{ij} is the signal coordinate vector and n=(x, y) consists of two independent Gaussian variables, each with zero-mean and variance σ_n^2.

The error probability of receiver can be written as a sum of error probabilities conditioned on the signal transmitted

$$P_e=\sum_{i=1}^{L_r}\sum_{j=1}^{L_i}P_e(O_{ij})P(O_{ij}), \tag{1}$$

where $P(O_{ij})=\int_{\theta_{i,j}}^{\theta_{i,j+1}}\int_{r_i}^{r_{i+1}}p(r,\phi)drd\phi$ is the probability of observed point with $p(r,\phi)=\dfrac{r}{2\pi\sigma^2}e^{-\frac{r^2}{2\sigma^2}}$ being the joint probability density function of the noise n in polar coordinates ($x=r\cos\phi$ and $y=r\sin\phi$) and $P_e(O_{ij})$ is the conditional error probability. L_r is the number of levels, L_i is the number of points on i-th level, θ_{ij} is the j-th decision phase on the i-th level, ϕ_{ij} is the j-th reconstruction phase on the i-th level (see Fig. 1.)

Since the region of integration is usually not trivial (see Fig. 2) bounds on the error probability are desirable. The most common approach to bounding P_e is to upper-bound each $P_e(O_{ij})$ using union bound [2]

$$P_e(O_{ij}) \le \sum_{m=1,\, m\ne i}^{L_r} \sum_{l=1, l\ne j}^{L_m} Q(\frac{d(O_{ij}, O_{ml})}{2\sigma_n}), \qquad (2)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy$ and $d(O_{ij}, O_{ml})$ is Euclidean distance between point $O_{i,j}$ and $O_{m,l}$.

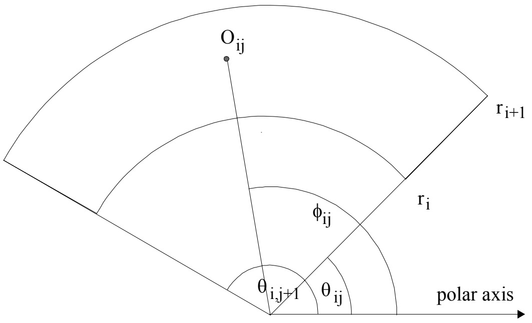

Fig .1. Quantization cell of representation point O_{ij}

The Hughes' upper bound is easy to calculate, can be applied to any signal constellation and is always better than union bound and minimum distance bound [3]

$$P_e(O_{ij}) \le N_{\min}(O_{ij}) \cdot B_2(\theta_0, \frac{d_{\min}(O_{ij})}{2\sigma_n}), \qquad (3)$$

where

$$d_{\min}(O_{ij}) = \min_{m\ne i,\, j\ne l} d(O_{ij}, O_{ml}),$$

$$B_2(\theta, x) = \frac{\theta}{\pi} - \frac{2\cot\theta}{\pi} G(\frac{x\tan\theta}{\sqrt{2}}, \cot\theta)$$

and θ_0 is determined from $B_2(\theta_0, 0) = 1/N_{\min}(O_{ij})$, with $N_{\min}(O_{ij})$ being the number of the nearest neighbours for observed point O_{ij}. The function $G(z, p)$ is defined as

$$G(z, p) = \int_0^z e^{-p^2 y^2} \int_0^y e^{-x^2} dx dy .$$

The Swaszek's lower bound is an asymptotically tight lower bound useful for small to medium values of SNR [4]

$$P_e(O_{ij}) \ge 5 \int_{d_{\min}(O_{ij})}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} [1 - 2Q(\frac{2w_i}{d_{\min}(O_{ij})\sigma})] dx \text{ ,} (4)$$

where $2w_i$ is the slab width as defined in [4].

Introduction of the polar coordinates ($x = r\cos\phi$, $y = r\sin\phi$) allow us to obtain a simple expression for exact error probability in 2D space. Conditional error probability in polar coordinates, starting from Fig. 2, can be determined as follows

$$P_e(O_{ij}) = \sum_{k=1}^{L_n(O_{ij})} \int_{\phi_k}^{\phi_{k+1}} \int_{\frac{d_m}{2\cos(\beta_k - \phi)}}^{\infty} \frac{r}{2\pi\sigma_n^2} e^{-\frac{r^2}{2\sigma_n^2}} dr d\phi =$$

$$= \sum_{k=1}^{L_n(O_{ij})} \frac{1}{2\pi} \int_{\phi_k}^{\phi_{k+1}} e^{-\frac{d_m^2(O_{ij}, O_{ij}^{(k)})}{8\sigma_n^2 \cos^2(\beta_k - \phi)}} d\phi, \qquad (5)$$

where d_m is defined bellow. ϕ_k is the angle between the polar axis and the segment of a line $O_{ij}P_k$, β_k is the angle between the polar axis and the segment of a line $O_{ij}O_{ij}^{(k)}$ (Fig. 2), $L_n(O_{ij})$ is the number of nearest neighbours having influence on decision region.

We can compute error probability for each constellation point if we determine the nearest neighbours and the decision regions around each point. Decision regions are irregular hexagons. Hexagon's sides are obtained at the straight lines crossing, ortogonally drown on line segments which connect the point under observation O_{ij} with the nearest neighbouring points $O_{ij}^{(k)}$ (see Fig. 2). Each of the straight lines are drown at the distance measuring $d_m(O_{ij}, O_{ij}^{(k)})/2$ from the observed point O_{ij}, where d_m is a modified Euclidean distance

$$\frac{d_m(O_{ij}, O_{ij}^{(k)})}{2} = \frac{d(O_{ij}, O_{ij}^{(k)})}{2} + $$

$$+ \frac{\sigma_n^2}{d(O_{ij}, O_{ij}^{(k)})} \ln\left[\frac{P(O_{ij})}{P(O_{ij}^{(k)})}\right]. \qquad (6)$$

$d(O_{ij}, O_{ij}^{(k)})$ is Euclidean distance between O_{ij} and $O_{ij}^{(k)}$ points, σ_n^2 is average noise power, $P(O_{ij})$, as we seen, is the probability of observed point, $P(O_{ij}^{(k)})$ is the probability of the neighbouring point. Modified Euclidean distance is derived starting from MAP principle of detection.

Now, the error probability can be obtained from the expression (1). So, the influence of any neighbour depends on the angle $(\phi_{k+1} - \phi_k)$, mutual Euclidean distance and mutual probabilities between points.

The proposed method can be applied for any nonuniform signal constellation and it is illustrated on the example of restricted iterative polar quantization (IPQ) [5].

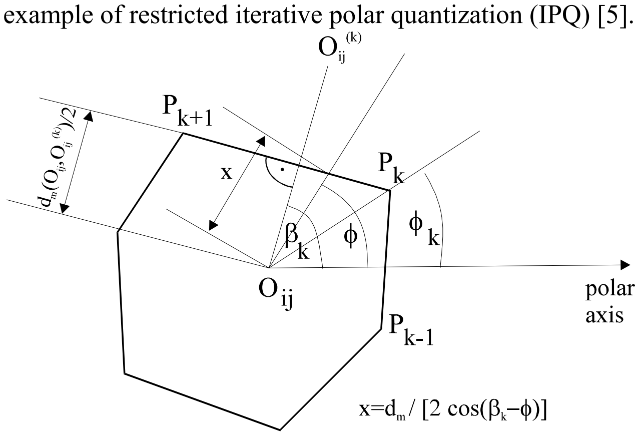

Fig. 2. A typical decision region for error probability determination

The restricted iterative polar quantization method presented in the paper [5] consists of a nonuniform scalar quantization of amplitude r and a uniform scalar quantization of phase ϕ and can be applied for any number of points. The signal constellation which is obtained after

IPQ of Gaussian source with decision regions for transmission through Gaussian channel (L=256) is shown in Fig. 3.

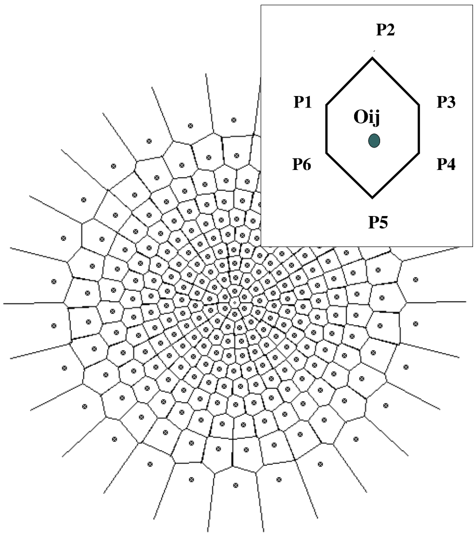

Fig. 3. The signal constellation with 256 points and decision regions

The error probability computed for signal constellation which is obtained by a nonuniform source iterative polar quantization as well as error probabilities per symbols for uniform signal constellation with 256 and 64 points are shown in Fig. 4.

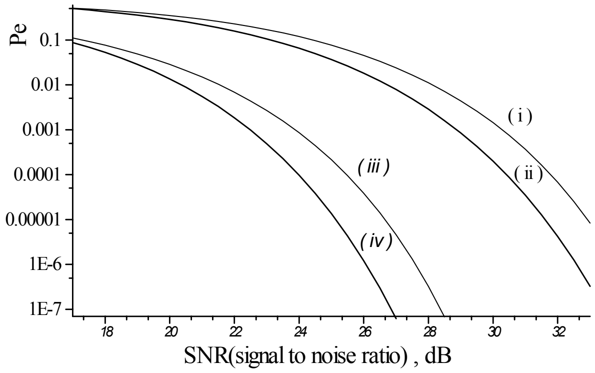

Fig. 4. The error probability per symbol for uniform and nonuniform signal constellation with 256 and 64 points. (i) uniform signal constellation for 256 points.(ii) nonuniform signal constellation for 256 points. (iii) uniform signal constellation for 64 points. (iv) nonuniform signal constellation for 64 points

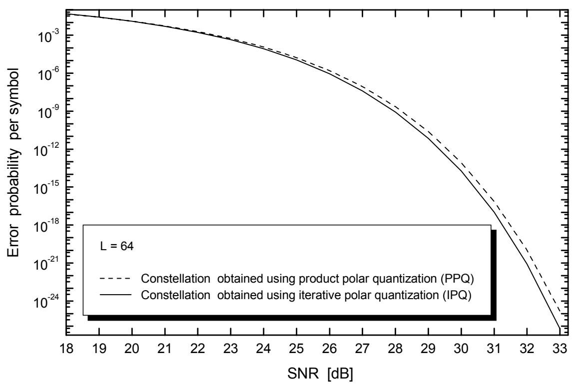

Fig. 5. Error probability per symbol for constellations obtained using PPQ and IPQ

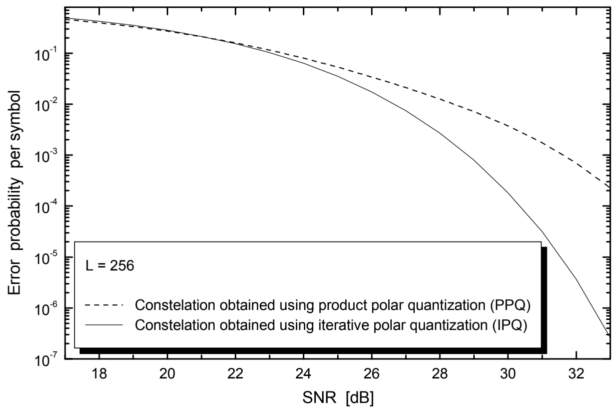

Fig. 6. Error probability per symbol for constellations obtained using PPQ and IPQ

The error probability per symbol for constellation obtained using product polar quantization (PPQ) [6] as well as iterative polar quantization (IPQ) [5] is shown in Figs. 5 and 6 for 64 and 256 constellation points, respectively.

Conclusion

The simple method for exact error probability determination of nonuniform signaling in two dimensions for Gaussian channel is presented in this paper. The error probability per symbol both for uniform constellations and nonuniform signal constellations obtained using different polar quantization methods with 256 and 64 points are accurately determined as illustration.

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Pateikiamas paprastas tikslios paklaidos tikimybės įvertinimo dvimačiame gausiniame kanale metodas. Paklaidos tikimybė randama naudojant maksimalios aposteorinės tikimybės nustatymo principą. Il.6, bibl.7 (anglų kalba; santraukos lietuvių, anglų ir rusų k.).

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A simple method for exact error probability determination of nonuniform signaling in two dimensions for Gaussian channel is presented in this paper. The error probability is determined using maximal aposteriori probability (MAP) detection principle. Ill. 6, bibl. 7 (in English; summaries in Lithuanian, English, Russian).

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