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Reliability Estimation when Failure Intensity Depends on Calendar Time

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Introduction

Suppose that X is a nonnegative random variable describing the failure time of a product. The reliability of products can be characterized by the reliability function $S(t) = \mathbf{P}\{X > t\}$, the failure rate $\lambda(t) = -S'(t)/S(t)$.

In many practical situations the failure rate of products depends not only on their age t but also on the date (calendar time) T. For example, at different dates the characteristics of spares of products, the technologic process, exploitation conditions may be different. Denote by $\lambda(t,T)$ the failure rate depending on the product age and calendar time.

Similar situations may be observed for other practical applications. For example, insurance companies: the mortality of individuals depends not only on their age but also on the calendar time (flu epidemic, accident rate and etc).

The Model

Suppose that we observe products (units) manufactured at the calendar time interval $[0, T^*]$. When a product is manufactured its age is $t = 0$. The products are observed at their age interval $[0, t^*]$. For example, t^* is the guarantee time.

Define the region restricted by lines $t = 0$, $t = t^*$, $T = t$, $T = T^*$ (see Fig. 1). In such region the run of a unit manufactured at the time moment $[0, T]$ can be plotted as a segment of line which forms 45° angle with t-axis. The failure is observed if the segment of line ends in the region. The circles and crosses denote different types of failure.

Suppose that a unit (manufactured at the moment $[0, T]$ and not failed at moment $[t, t+T]$) fails because of one of r possible failure types with probability

$$\lambda^{(s)}(t,T)\Delta t + o(\Delta t), \quad \Delta t \to 0, \quad s = 1,\dots,r. \qquad (1)$$

Set $\lambda(t,T) = \sum_s \lambda^{(s)}(t,T)$, where s denotes the failure type.

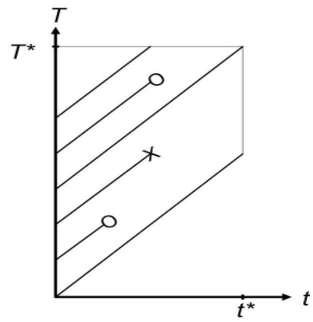

Fig. 1. The flow of products

The cumulative failure rate $\Lambda(t, t+T)$ can be written in the form:

$$\Lambda(t, t+T) = h(T)\int_0^t \delta(x)dx + \int_T^{T+t} g(x)dx, \qquad (2)$$

where $\delta(x)$ is related to the unit age, $h(T)$ to the variation of technological processes, $g(T)$ to the variation of exploitation conditions (for example, the influence of seasonal climate variation).

Suppose that the form of functions h, δ, g is known and they depend on unknown parameters. Then standard parametric methods can be applied.

In many practical situations the selection of appropriate functions form is complicated. Then non-parametric or semiparametric methods are used.

The estimation of hazard rates and various reliability characteristics using non-parametric and semiparametric methods are discussed in [2], [3].

The statistical data can be in the form which requires the parametrization of model. For example we know only the number of manufactured units at particular calendar time intervals and the number of failed units at particular age intervals but the moments of manufacturing or failure are unknown.

Suppose that the time interval $(0, t^*)$ is partitioned into m equal subintervals. Denote by τ the length of sub-interval. So $m\tau = t^*$. Suppose that the interval $(0, T^*)$ is partitioned into k subintervals of the same length. So $k\tau = T^*,\ k > m$ (see Fig. 2).

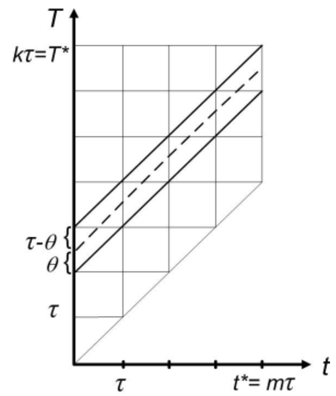

Fig. 2. The grouping intervals

Suppose that n_j units were manufactured at calendar time interval $[(j-1)\tau,\ j\tau]$. At the first subinterval $0 \le t \le \tau$ the part of units can fail at the calendar time interval $[(j-1)\tau,\ j\tau]$, and another part of units at the calendar time interval $[j\tau, (j+1)\tau]$. Denote respectively by Z_{1j} and $\tilde{Z}_{1j}$ these quantities of units. Similarly define quantities of failed units at the interval $(i-1)\tau \le t \le i\tau$. Denote them by Z_{ij} and $\tilde{Z}_{ij}$, $i = 1, \ldots, \nu_j$, $\nu_j = \min(m, k - j + 1)$; $\tilde{Z}_{\nu_j, k} = 0$, if $\nu_j < m$;

$$Z_{ij} = \sum\nolimits_s Z_{ij}^{(s)},\ \tilde{Z}_{ij} = \sum\nolimits_s \tilde{Z}_{ij}^{(s)}; \tag{3}$$

where s is the failure type.

For such data the parametrization can be used. Suppose that failure intensity piecewise constant functions are

$$\lambda^{(s)}(t, T) \equiv \lambda_{ij}^{(s)},\ (i-1)\tau \le t \le i\tau,\ (j-1)\tau \le T \le j\tau; \tag{4}$$

$$\sum_{s=1}^{r} \lambda_{ij}^{(s)} = \lambda_{ij},\ i = 1, \ldots, \nu_j,\ j = 1, \ldots, k.$$

So we obtain the parametric model with the parameter

$$\boldsymbol{\lambda} = (\lambda_{ij}^{(s)},\ s = 1, \ldots, r,\ i = 1, \ldots, \nu_j,\ j = 1, \ldots, k). \tag{5}$$

Let us obtain the likelihood function. Suppose that a unit was manufactured at the time moment (see Fig. 2) $T = (j-1)\tau + \theta,\ 0 \le \theta \le \tau$. At the first age subinterval $0 \le t \le \tau$ the probability of such unit failure is

$$p_{1j}(\theta) = 1 - e^{-(\tau-\theta)\lambda_{1j}}, \tag{6}$$

if $(j-1)\tau \le T \le j\tau$, and is

$$\tilde{p}_{1j}(\theta) = e^{-(\tau-\theta)\lambda_{1j}}(1 - e^{-\theta\lambda_{1,j+1}}), \tag{7}$$

if $j\tau \le T \le (j+1)\tau$.

Similarly at the age interval $(i-1)\tau \le t \le i\tau$ these probabilities:

$$\begin{cases} p_{1j}(\theta) = e^{-\Lambda_{1j}(\theta)}(1 - e^{-(\tau-\theta)\lambda_{ij}}), \\ \tilde{p}_{1j}(\theta) = e^{-\Lambda_{ij}(\theta)-(\tau-\theta)\lambda_{ij}}(1 - e^{-\theta\lambda_{ij}}); \end{cases} \tag{8}$$

where $\Lambda_{ij}(\theta)$ is the cumulative failure intensity function until an age moment $t = (i-1)\tau$:

$$\Lambda_{ij}(\theta) = (\theta - \tau)\sum_{l=1}^{i-1} \lambda_{l, j+l-1} + \theta \sum_{l=1}^{i-1} \lambda_{l, j+l}. \tag{9}$$

The probability of a unit surviving is

$$q_j(\theta) = 1 - \sum\nolimits_i (p_{ij}(\theta) + \tilde{p}_{ij}(\theta)). \tag{10}$$

Denote by U_j the quantity of non-failure units. So

$$\sum\nolimits_i (Z_{ij} + \tilde{Z}_{ij}) + U_j = n_j. \tag{11}$$

If manufacturing is stable then it is natural to suppose that the manufacturing moment θ of a unit is uniformly distributed at the interval $(0, \tau)$. Taking expectation with respect to this distribution we obtain the failure probabilities of a unit manufactured at the moment $T \in [(j-1)\tau,\ j\tau]$:

$$\begin{cases} p_{ij} = p_{ij}(\boldsymbol{\lambda}) = \dfrac{1}{\tau}\displaystyle\int_0^\tau p_{ij}(\theta)d\theta,\ \tilde{p}_{ij} = \tilde{p}_{ij}(\boldsymbol{\lambda}) = \dfrac{1}{\tau}\displaystyle\int_0^\tau \tilde{p}_{ij}(\theta)d\theta; \\ q_j = q_j(\boldsymbol{\lambda}) = 1 - \sum\nolimits_i (p_{ij}(\boldsymbol{\lambda}) + \tilde{p}_{ij}(\boldsymbol{\lambda})). \end{cases} \tag{12}$$

The distribution of the vector $(Z_{1j}, \tilde{Z}_{1j}, \ldots, Z_{\nu_j, j}, \tilde{Z}_{\nu_j, j})$ is polynomial distribution

$$P_{2\nu_j}(n_j; (p_{ij}, \tilde{p}_{ij}, i = 1, \ldots, \nu_j, q_j)). \tag{13}$$

Parameter Estimators

Let $\gamma_{ij}^{(s)} = \lambda_{ij}^{(s)} / \lambda_{ij}$. Then the likelihood function of the vector $(Z_{1j}^{(s)}, \tilde{Z}_{1j}^{(s)}, \ldots, Z_{\nu_j, j}^{(s)}, \tilde{Z}_{\nu_j, j}^{(s)})$, $s = 1, \ldots, r$ is

$$L_j(\boldsymbol{\lambda}) = C\prod_{i=1}^{\nu_j}\left\{\prod_{s=1}^{r}[\gamma_{ij}^{(s)}]^{Z_{ij}^{(s)} + \tilde{Z}_{ij}^{(s)}}\ p_{ij}^{Z_{ij}}\ \tilde{p}_{ij}^{\tilde{Z}_{ij}}\right\} q_j^{U_j} \tag{14}$$

and their multiplication is

$$L(\boldsymbol{\lambda}) = \prod_{j=1}^{k} L_j(\boldsymbol{\lambda}). \quad (15)$$

We obtain maximum likelihood estimators

$$\hat{\gamma}_{ij}^{(s)} = \frac{Z_{ij}^{(s)} + \tilde{Z}_{ij}^{(s)}}{Z_{ij} + \tilde{Z}_{ij}}, \quad \hat{\lambda}_{ij}^{(s)} = \hat{\gamma}_{ij}^{(s)} \hat{\lambda}_{ij}. \quad (16)$$

The complicated system

$$\frac{\partial \ln L}{\partial \lambda_{ij}} = 0, \ i = 1,\dots,\nu_j, \ j = 1,\dots,k \quad (17)$$

for maximum likelihood estimators $\hat{\lambda}_{ij}$ is solved by numerical methods.

The Goodness-of-fit Test

The chi-square goodness-of-fit test can be used to assess whether the model is consistent with the data. If the hypothesis is true then the distribution of the test statistic (see [1])

$$X^2 = \sum_{j=1}^{k} \left[\sum_{i=1}^{\nu_j} \sum_{s=1}^{r} \left\{ \frac{(Z_{ij}^{(s)} - n_j \hat{\gamma}_{ij}^{(s)} p_{ij}(\hat{\boldsymbol{\lambda}}))^2}{n_j \hat{\gamma}_{ij}^{(s)} p_{ij}(\hat{\boldsymbol{\lambda}})} + \frac{(\tilde{Z}_{ij}^{(s)} - n_j \hat{\gamma}_{ij}^{(s)} p_{ij}(\hat{\boldsymbol{\lambda}}))^2}{n_j \hat{\gamma}_{ij}^{(s)} p_{ij}(\hat{\boldsymbol{\lambda}})} \right\} + \frac{(U_j - n_j q_j(\hat{\boldsymbol{\lambda}}))^2}{n_j q_j(\hat{\boldsymbol{\lambda}})} \right] \quad (18)$$

is approximately (for sufficiently large n_j, $j = 1,\dots,k$) chi-square distribution with $\kappa = sm(k - (m+1)/2)$ degrees of freedom (the number of random variables $Z_{ij}, \tilde{Z}_{ij}$ is $sm(2k - m)$ and the number of estimated parameters is $sm(k - (m-1)/2)$). The hypothesis is rejected with significance level α if $X^2(\hat{\boldsymbol{\lambda}}) > \chi_\alpha^2(\kappa)$, where $\chi_\alpha^2(\kappa)$ is α critical value of the chi-squre distribution with κ degrees of freedom.

The chi-square statistic can be used to test the hypothesis that some parameters are the same. For such hypothesis the estimator $\tilde{\boldsymbol{\lambda}}$ of the parameter vector $\boldsymbol{\lambda}$ is obtained under assumption that hypothesis is true. Suppose that the dimension of parameter vector decreases in quantity l. If the hypothesis is true then the distribution of the test statistic $X^2(\tilde{\boldsymbol{\lambda}})$ (see (5)) is chi-square with $sm(2k - m) - l$ degrees of freedom.

Example. Let us consider the data of reclamations of the kinescope manufacturing enterprise. The data is for two years, grouped by quarters (see Table 1).

The column j represents quarters. The column n_j represents numbers of manufactured kinescopes in that quarter.

Table 1. The data

j	n_j	$Y_{1j}^{(1)};Y_{2j}^{(1)}$	$Y_{1j}^{(2)};Y_{2j}^{(2)}$	$Y_{1j}^{(3)};Y_{2j}^{(3)}$	$Y_{1j}^{(4)};Y_{2j}^{(4)}$
1	385 895	144; 31	85; 19	69; 15	519; 131
2	434 300	173; 48	99; 27	54; 14	962; 240
3	377 437	159; 36	107; 25	45; 9	913; 229
4	574 933	265; 73	146; 37	67; 19	769; 191
5	509 579	212; 53	109; 31	27; 7	653; 162
6	560 007	256; 58	182; 42	33; 8	1234; 306
7	606 227	260; 62	147; 41	36; 6	1365; 341
8	780 897	278; 78	209; 49	54; 16	1061; 266
Σ	4 229 275	1747; 434	1084; 271	385; 94	7476; 1866

The next four columns correspond to different failure types. In each column two numbers correspond to reclamations collected in the first and the second month after the unit was manufactured. The produced kinescopes go to TV manufacturing enterprise from which the reclamations are obtained during two months only.

The first group consists of electrical parameters failures (for example, small beam current, low voltage). The second failure group includes the vacuum level failures. The third failure group is parasitic emission and voltage outflow. The most reclamations are from the fourth group of failures which are mechanical damages (for example, scratches, damaged contacts) and dust or other particles in the kinescopes.

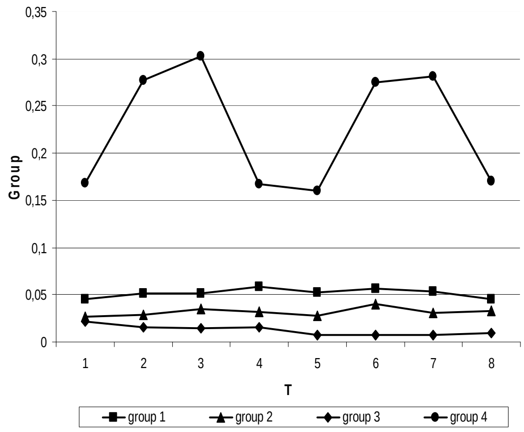

Fig. 3. The variation of four types of reclamations

The graph shows that the first and the second group of reclamations percentage level are almost the same. The percent of the third group reclamations is smallest in the second year. The percent of the fourth group of reclamations is much higher in the second and the third quarters. The explanation is that in the spring and the summer there are more dust in the air and because of that kinescopes have more particles inside.

The data are grouped. For example, $Y_{1j}^{(s)}$ is the sum:

$$Y_{1j}^{(s)} = \sum_{i=1}^{4} (Z_{ij}^{(s)} + \tilde{Z}_{ij}^{(s)}), \ j = 1,\dots,8. \quad (19)$$

Set $p_{ij}^{(s)} = \mathbf{P}\{Y_{ij}^{(s)} = 1\}$, $i = 1,2$; $j = 1,\dots,8$; $s = 1,\dots,4$.

Then the dimension of unknown parameters $p_{ij}^{(s)}$ vector is the same as the dimension of observed random vector $(Y_{ij}^{(s)},\ \ i=1,2;\ \ j=1,\ldots,8;\ \ s=1,\ldots,4)$. The estimators of parameters $p_{ij}^{(s)}$ are frequencies $\hat{p}_{ij}^{(s)} = Y_{ij}^{(s)} / n_j$.

The proposed model for that data depends on 64 parameters ($k \times m \times r = 8 \times 2 \times 4$). After testing the hypothesis about parameters only 7 parameters are significant: two parameters describe the failure intensity for the first and the second group of parameters (one parameter for each group; total estimates of failure intensities are 0,052% and 0,032%, respectively); two parameters for the second parameter group (the first and the second year; estimates are 0,017% and 0,008%, respectively); two parameters for the fourth parameter group (1-4 and 2-3 quarters; estimates are 0,017% and 0,008%, respectively); the parameter δ shows how many times the failure intensity is smaller in the second month (after the kinescope was manufactured) than in the first month; the estimate is $\hat{\delta} = 0{,}25$. If the hypothesis is true the test statistic has approximately chi-square distribution with 57 degrees of freedom. The hypothesis is not rejected: the value of the test statistic is 64,2. Further reduction of the number of parameters is impossible.

The proposed model is suitable for the analysis of this data type. In the first stage we had a lot of parameters but after the model analysis we got minimal number of parameters which have clear meaning.

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In many practical situations the failure rate of products depends not only on their age but also on the date (calendar time). For example, at different dates the characteristics of spares of products, the technologic process, exploitation conditions may be different. Similar situations may be observed for other practical applications. For example, insurance companies: the mortality of individuals depends not only on their age but also on the calendar time (flu epidemic, accident rate and etc). The problem of failure rate estimation was investigated when the failure rate of products depends not only on their age but also on the date (calendar time) and statistical data are grouped into some intervals of product age and calendar time. The mathematical model was proposed. The parameter estimators were obtained. The test of model consistency was constructed. Ill. 3, bibl. 3 (in English; summaries in English, Russian and Lithuanian).

Ю. Круопис, Р. Левулене. Оценка надежности, когда интенсивность отказов зависит от календарного времени // Электроника и электротехника. – Каунас: Технология, 2009. – № 5(93). – С. 35–38.

Во многих практических применениях интенсивность отказов изделий зависит не только от их возраста, но и от календарного момента его изготовления. Это можно объяснить, например, изменениями характеристик сырья, нестабильностью технологического процесса, изменениями условий эксплуатации и т.п. С аналогичными проблемами сталкиваются и в других областях, например, страховые компании: смертность индивидуумов зависит не только от их возраста, но и от календарного времени (эпидемии гриппа, изменения аварийности и т.п.). Рассматривается задача построения оценок интенсивности отказов изделий в ситуации, когда функция интенсивности отказов зависит не только от возраста изделия, но и от календарного момента его изготовления, а статистические данные сгруппированы по некоторым интервалам возрастного и календарного времени. Построена математическая модель, найдены оценки параметров и представлены критерии для оценки адекватности модели. Ил. 3, библ. 3 (на английском языке; рефераты на английском, русском и литовском яз.).

J. Kruopis, R. Levulienė. Patikimumo vertinimas, kai gedimų intensyvumas priklauso nuo kalendorinio laiko // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2009. – Nr. 5(93). – P. 35–38.

Praktikoje daugeliu atvejų gaminių patikimumas priklauso ne tik nuo jų amžiaus, bet ir nuo pagaminimo kalendorinio laiko momento. Tai galima paaiškinti, pavyzdžiui, žaliavų charakteristikų svyravimais, technologinio proceso pakeitimais, eksploata-cinių sąlygų kitimu ir pan. Su analogiška situacija susiduriama ir kitose srityse, pavyzdžiui, draudime: individų mirtingumo intensyvumas priklauso ne tik nuo jų amžiaus bet ir nuo kalendorinio laiko (gripo epidemijos, avaringumo keliuose kitimas ir pan.). Darbe nagrinėjamas gaminių gedimų intensyvumo funkcijos įvertinių radimo uždavinys, kai ši funkcija priklauso ne tik nuo gaminio amžiaus, bet ir nuo jo pagaminimo kalendorinio laiko momento, o statistiniai duomenys sugrupuoti tam tikruose gaminio amžiaus ir kalendorinio laiko intervaluose. Sudarytas matematinis modelis, gauti parametrų įvertiniai ir pateikiami kriterijai modelio adekvatumui tikrinti. Il. 3, bibl. 3 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).